DECOMPOSITION OF LAGUERRE POLYNOMIALS WITH RESPECT TO THE CYCLIC GROUP OF ORDER n

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Abstract

Let n be an arbitrary positive integer. We decompose the Laguerre polynomials $L_m^{(\alpha)}$ as the sum of n polynomials $L_m^{(\alpha,n,k)}$; $m \in \mathbb{N}$; $k = 0, 1, ..., n-1$; defined by

$$L_m^{(\alpha,n,k)}(z) = \frac{1}{n} \sum_{l=0}^{n-1} \exp\left(-\frac{2i\pi kl}{n}\right) L_m^{(\alpha)}\left(z \exp\left(\frac{2i\pi l}{n}\right)\right); \quad z \in C.$$

In this paper, we establish the close relation between these components and the Brafman polynomials. The use of a technique described in an earlier work [2] leads us firstly to derive, from the basic identities and relations for $L_m^{(\alpha)}$, other analogous for $L_m^{(\alpha,n,k)}$ that turn out to be two integral representations, an operational representation, some generating functions defined by means of the generalized hyperbolic functions of order n and the hyper-Bessel functions, some finite sums including multiplication and addition formulas, a $(2n+1)$-term recurrence relation and a differential equation of order $2n$. Secondly, to express some identities of $L_m^{(\alpha)}$ as functions of the polynomials $L_m^{(\alpha),n,k}$. Some particular properties of $L_m^{(\alpha,n,0)}$, the first component, will be pointed out.

1   Introduction

It is assumed that the reader is acquainted with the results reported in an earlier work [2]. The notations and terminologies used in this paper will be continued. In particular, the reader is reminded that $\Omega(I) \equiv \Omega$ denotes the space of complex functions admitting a Laurent expansion in an annulus I with center in the origin and for an arbitrary positive integer n, every function f in Ω can be written as the sum of n functions $f_{[n,k]}$; $k = 0, 1, ..., n-1$; defined by (cf. Ricci [19] p.44 Eq.(3.3)):

$$f_{[n,k]}(z) = \frac{1}{n} \sum_{l=0}^{n-1} \omega_n^{-kl} f(\omega_n^l z); \quad z \in \mathbb{I},$$

with $\omega_n = \exp\left(\frac{2i\pi}{n}\right)$ the complex n-root of unity.

This paper deals with the case of Laguerre polynomials $L_m^{(\alpha)}$, we shall study in detail the polynomials

$$L_m^{(\alpha,n,k)}(z) = \frac{1}{n} \sum_{l=0}^{n-1} \omega_n^{-kl} L_m^{(\alpha)}(\omega_n^l z); \quad z \in \mathbb{C}. \tag{1.1}$$

With the two additional parameters n and k these polynomials can be viewed as generalizations of the polynomials $L_m^{(\alpha)}$ so we begin by situating the components $L_m^{(\alpha,n,k)}$ among the generalizations of $L_m^{(\alpha)}$ in the literature, more precisely, we shall state a relation between these components and the Brafman polynomials. Thereafter we use some results established in [2] to derive from the basic identities and relations for $L_m^{(\alpha)}$, other analogous for $L_m^{(\alpha,n,k)}$. More precisely, we shall state for these components two integral representations, an operational representation, some generating functions, a $(2n+1)$-term recurrence relation, a differential equation of order $2n$, some finite sums including multiplication and addition formulae. The converse of $L_m^{(\alpha,n,0)}$, the first component, will be classified according to some known families in the literature.

2   *Representation as Hypergeometric Series*

The action of the projection operator $\prod_{[n,k]}$ on both sides of the following representation (see for instance [26], p.99 Eq.(5.3.3)):

$$L_m^{(\alpha)}(z) = \frac{(\alpha+1)_m}{m!} F_1 \left(\begin{array}{c} -m \\ \alpha+1 \end{array} ; z \right),$$

considered as functions of the variable z and the use of the Osler-Srivastava identity (cf.[16], p.890 Eq.(5) or [24], p.194 Eq.(12)) give rise to the relation

$$L_m^{(\alpha,n,k)}(z) = \frac{\Gamma(m+\alpha+1)(-m)_k}{\Gamma(m+\alpha+1)m!k!} z^k \times$$

$$\times \; _nF_{2n-1} \left(\begin{array}{c} \Delta(n,-m+k) \\ \Delta^*(n,k+1), \Delta(n,\alpha+k+1) \end{array} ; \left(\frac{z}{n}\right)^n \right), \tag{2.1}$$

where $\Delta(n,\lambda)$ is the set of n parameters:

$$\Delta(n,\lambda) \equiv \left\{ \frac{\lambda}{n}, \frac{\lambda+1}{n} \ldots, \frac{\lambda+n-1}{n} \right\}; \quad n \in \mathbb{N}^*,$$

and $\Delta^*(n,k+1) = \Delta(n,k+1)\backslash\{\frac{n}{n}\}$.
It is possible to express these components by the Brafman polynomials

defined by (cf.[7], p.186 Eq.(52)):

$$\mathcal{B}_m^n[a_1, \ldots, a_r; b_1 \ldots, b_s; z] =_{n+r} F_s \left(\begin{array}{c} \Delta(n, -m), a_1, \ldots, a_r \\ b_1 \ldots, b_s \end{array} ; z \right), \quad (2.2)$$

where $(n, m) \in \mathbb{N}^* \times \mathbb{N}$ and the complex parameters a_i; $i = 1, \ldots, r$; and b_j; $j = 1, \ldots, s$; are independent of m and z. We have in fact

$$L_m^{(\alpha, n, k)}(z) =$$

$$= \frac{\Gamma(m + \alpha + 1)(-m)_k}{\Gamma(m + \alpha + 1)m!k!} z^k B_{m-k}^n[-; \Delta^*(n, k+1), \Delta(n, \alpha + k + 1); \left(\frac{z}{n}\right)^n];$$

$$k \le m. \tag{2.3}$$

3   *Operational Representation*

We begin by expressing the Laguerre polynomials $L_m^{(\alpha)}(z)$ by functions belonging to Ω and homogeneous operators (cf.[2], section III for definition). Recall that the Laguerre polynomials has the well known following operational representation (see for instance [11] p.188 Eq.(5)):

$$L_m^{(\alpha)}(z) = \frac{e^z z^{-\alpha}}{m!} D^m(z^{\alpha+m} e^{-z}); \quad D \equiv \frac{d}{dz}. \tag{3.1}$$

One can justify by induction the following identity

$$z^{-\alpha} D^m(z^{\alpha+m} f) = \prod_{j=1}^{m} (zD + \alpha + j)f, \tag{3.2}$$

where f is an arbitrary differentiable function of z. We have then

$$L_m^{(\alpha)}(z) = \frac{e^z}{m!} (zD + \alpha + 1)_m \{e^{-z}\}, \tag{3.3}$$

where

$$(zD + \alpha + 1)_m = \prod_{j=1}^{m} (zD + \alpha + j).$$

Now, according to the decomposition (1.2) in [2], the operators $(zD + \alpha + 1)$ and $(zD + \alpha + 1)_m$, are homogeneous of degree zero. Then, if we apply the projection operators $\prod_{[n,k]}$ to the two members of (3.3) and we use the theorem III-1 and the corollary II-2 in [2] we obtain

$$L_m^{(\alpha, n, k)}(z) = \frac{1}{m!} \sum_{p=0}^{n-1} h_{n,p}(z)(zD + \alpha + 1)_m \{h_{\underbrace{n, k-p}}(-z)\}, \tag{3.4}$$

or, equivalently,

$$L_m^{(\alpha,n,k)}(z) = \frac{1}{m!} \sum_{p=0}^{n-1} h_{n,p}(z) z_{-\alpha} D^m \left(z^{\alpha+m} h_{\underbrace{n,\,k-p}}(-z) \right), \qquad (3.5)$$

where $h_{n,k}$ is the hyperbolic function of order n and k-th kind defined by (see for instance [12], p.213 Eq.(8)):

$$h_{n,k}(z) = \sum_{m=0}^{\infty} \frac{z^{nm+k}}{(nm+k)!}.$$

In particular, when $n = 2$ and $k = 0$, we have

$$L_m^{(\alpha,2,0)} = \frac{1}{m!}(\cosh z) z^{-\alpha} D^m (z^{\alpha+m} \cosh z) - \sinh z z^{-\alpha} D^m (z^{\alpha+m} \sinh z)).$$

4   Integral Representation

From the Koshlyakov's formula (cf.[17], p.94 or [16], p.155 Eq.(14)):

$$L_m^{(\alpha+\beta)}(x) = \frac{\Gamma(m+\alpha+\beta+1)}{\Gamma(\beta)\Gamma(m+\alpha+1)} \int_0^1 t^\alpha (1-t)^{\beta-1} L_m^{(\alpha)}(xt) dt;$$

$$\alpha > -1, \quad \beta > 0 \qquad (4.1)$$

we obtain, after application of the projection operator $\prod_{[n,k]}$ to the two members of (4-1) considered as functions of the variable x, the following integral representation (4-2)

$$L_m^{(\alpha+\beta,n,k)}(x) =$$

$$\frac{\Gamma(m+\alpha+\beta+1)}{\Gamma(\beta)\Gamma(m+\alpha+1)} \int_0^1 t^\alpha (1-t)^{\beta-1} L_m^{(\alpha,n,k)}(xt) dt; \qquad (4.2)$$

$$\alpha > -1, \quad \beta > 0.$$

Notice that this formula can be justified by another way using the following identity (cf.[18], p.104 Eq.(5)))

$$_rF_{s+n} \left(\begin{array}{cc} (a_r) & \\ (b_{s,}) & \Delta(n,\alpha+\beta) \end{array} ; x \right) =$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \,_rF_{s+n} \left(\begin{array}{cc} (a_r) & \\ (b_{s,}) & \Delta(n,\alpha) \end{array} ; xt^n \right).$$

Another integral representation can be established by applying the proposition IV-1 in [2], that is

$$L_m^{(\alpha,n,k)}(x) = \frac{1}{2i\pi} \int\limits_{|s|=R} \frac{s^{n-1-k}z^k}{s^n - z^n} L_m^{(\alpha)}(s)ds; \quad |z| < R, \qquad (4.3)$$

or, equivalently,

$$L_m^{(\alpha,n,k)}(re^{i\theta}) = \int\limits_0^{2\pi} P_{n,k}(R,r,\phi-\theta)L_m^{(\alpha)}(Re^{i\phi})d\phi$$

$$= \int\limits_0^{2\pi} P_{n,k}(R,r,\phi-\theta)L_m^{(\alpha,n,k)}(Re^{i\phi})d\phi; \quad r < R,$$

with

$$P_{n,k}(R,r,\phi-\theta) = \left\{ \left(R^{2(n-k)} - r^{2(n-k)} \right) R^k r^k e^{-ik(\phi-\theta)} + \right.$$

$$\left. \left(R^{2k} - r^{2k} \right) R^{n-k} r^{n-k} e^{-i(n-k)(\phi-\theta)} \right\} /2\pi(R^{2n} - r^{2k} - 2R^n r^n \cos n(\phi-\theta)).$$
$$(4.4)$$

For $n = 1$, this integral representation specializes to Poisson integral formula.

We remark in passing that among the consequences of the results in the next section concerning the generating functions for $L_m^{(\alpha,n,k)}$ there is the possibility of obtaining corresponding integral representations of Cauchy type.

5   *Generating Functions*

A very large number of generating functions for Laguerre polynomials $L_m^{(\alpha)}$ are known. We recall below the more important, or more useful of them (see for instance [11], p.189):

$$\sum_{m=0}^{\infty} L_m^{(\alpha)}(x)t^m = (1-t)^{-\alpha-1} \exp\left(\frac{-xt}{1-t} \right); \quad |t| < 1; \qquad (5.1)$$

$$\sum_{m=0}^{\infty} L_m^{(\alpha-m)}(x)t^m = (1+t)^{\alpha} e^{-xt}; \quad |t| < 1; \qquad (5.2)$$

$$\sum_{m=0}^{\infty} \frac{t^m}{(\alpha+1)_m} L_m^{(\alpha)}(x) = e^t \ _0F_1\left(\begin{array}{c} - \\ \alpha+1 \end{array}; -xt\right) = e^t j_\alpha\left(2\sqrt{xt}\right). \qquad (5.3)$$

Now, from all the above generating functions for Laguerre polynomials $L_m^{(\alpha)}$ corresponding ones for $L_m^{(\alpha,n,k)}$ can be obtained by a mere mechanical application of the projection operators $\prod_{[n,k]}$ to the two members of each relations. Thus, we obtain

$$\sum_{m=0}^{\infty} L_m^{(\alpha,n,k)}(x) t^m = \frac{1}{(1-t)^{\alpha+1}} h_{n,k}\left(\frac{xt}{t-1}\right); \quad |t| < 1; \qquad (5.4)$$

$$\sum_{m=0}^{\infty} L_m^{(\alpha-m,n,k)}(x) t^m = (1+t)^\alpha h_{n,k}\left(-xt\right); \qquad (5.5)$$

$$\sum_{m=0}^{\infty} \frac{t^m}{(\alpha+1)_m} L_m^{(\alpha,n,k)}(x) = e^t \left(j_\alpha\right)_{[2n,2k]}\left(2\sqrt{xt}\right), \qquad (5.6)$$

where $(j_\alpha)_{[2n,2k]}$ is defined by the identity (2.4) in [3].

$$(j_\alpha)_{[2n,2k]}(x) =$$

$$= \frac{1}{k!\,(\alpha+1)_k}\left(\frac{iz}{2}\right)^{2k} \ _0F_{2n-1}\left(\begin{array}{c} - \\ \Delta^*(n,k+1), \Delta(n,\alpha+1) \end{array}; \left(\frac{iz}{2n}\right)^{2n}\right),$$

which can be expressed by the hyper-Bessel functions of order $2n$ and index $\Delta^*(n,k+1), \quad \Delta(n,\nu+1)$ introduced by Delerue [8].

Notice that these formulae can be justified by other ways using some identities already established in the literature. Thus: for instance for $k = 0$:

(i) the formula, (5.4) follows from the Brafman identity (cf.[7], p.186 Eq.(55)):

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} \mathcal{B}_m^n\left[(\alpha_r);(\beta_s);x\right] \cdot t^m =$$

$$= (1-t)^{-\lambda} \ _{n+r}F_s\left(\begin{array}{c} \Delta\left(n,\lambda\right), \ (\alpha_r) \\ (\beta_s) \end{array}; x\left(\frac{t}{t-1}\right)^n\right)$$

on setting $r = 0, s = 2n - 1$ and $(\beta_s) \equiv \Delta^*(n,1) \cup \Delta(n,\alpha+1)$.

(ii) the formula (5.5) follows from the Srivastava-Buschman's identity (cf.[25] p.364 Eq.(17)):

$$\sum_{m=0}^{\infty} \binom{\alpha + m}{m} \times$$

$$\times \; _{r+q}F_{s+q} \left(\begin{array}{cc} (a_r), & \Delta(q, -m) \\ (b_s), & \Delta(q - p, 1 + \alpha), \end{array} \Delta(p, -\alpha - m) \; ; \frac{q^q x}{(-p)^p (q - p)^{q-p}} \right) \cdot t^m$$

$$= (1 - t)^{-\alpha - 1} \; _rF_s \left(\begin{array}{c} (\alpha_r) \\ (\beta_s) \end{array} ; \frac{x(-t)^q}{(1 - t)^{q-p}} \right),$$

where p is a positive integer less or equal to q, on setting $p = q = n, s = n - 1, r = 0$ and $(\beta_s) \equiv \Delta^*(n, 1)$.

(iii) the formula (5.6) follows from the Srivastava's identity (cf.[20], p.203 Eq.(8) or [22], p.68 Eq.(3.9)):

$$\sum_{m=0}^{\infty} \; _{n+r}F_s \left(\begin{array}{cc} \Delta(n, -m), & (\alpha_r) \\ & (\beta_s) \end{array} ; x \right) \cdot \frac{t^m}{m!} = e^t \; _rF_s \left(\begin{array}{c} (\alpha_r) \\ (\beta_s) \end{array} ; x \left(\frac{-t}{n} \right)^n \right)$$

on setting $r = 0, s = 2n - 1$ and $(\beta_s) \equiv \Delta^*(n, 1) \cup \Delta(n, \alpha + 1)$.

6   Recurrence Relation

In this section, we shall establish a general result for orthogonal polynomials. Thereafter, we shall consider the particular case of Laguerre polynomials.

Let $\{a_m\}_{m=0}^{\infty}$ and $\{b_m\}_{m=0}^{\infty}$ be real sequences with $b_m \neq 0$. Let $\{P_m(x)\}_{m=0}^{\infty}$ be a sequence of polynomials satisfying the recurrence formula :

$$xP_m(x) = b_{m-1}P_{m-1}(x) + a_m P_m(x) + b_m P_{m+1}(x); \quad m \geq 0, \qquad (6.1)$$

with $P_{-1}(x) = 0$ and $P_0(x) = 1$.

The Jacobi matrix or J-matrix associated with $\{P_m(x)\}_{m=0}^{\infty}$ is the following real infinite matrix:

$$J = (\alpha_{ij})_{i,j=0,1,2,\ldots}.$$

where the coefficients (α_{ij}) are defined by

$$\alpha_{i\;i} = a_i; \alpha_{i+1\;i} = a_{i\;i+1} = b_i \;\; and \;\; a_{i\;j} = 0 \;\; if \;\; |i - j| > 1.$$

For a natural integer r, the matrix J^r is a band symmetric matrix with $(2r + 1)$ diagonals, that is to say

$$J^r = \left(\alpha_{ij}^{(r)} \right)_{i,j=0,1,2,\dots} \quad with \quad \alpha_{i\,j}^{(r)} = 0 \;\; if \;\; |i - j| > r.$$

Now, if we multiply both sides of (6.1) by the variable x and we use (6.1) to eliminate x in the right side, we obtain a recurrence relation of order four satisfied by the polynomials $\{P_m(x)\}_{m=0}^{\infty}$. The reiteration of this process $(r - 1)$ times gives rise to the following recurrence relation:

$$x^r P_m(x) = \sum_{j=\sup(-m,-r)}^{r} \alpha_{m\ m+j}^{(r)} \; P_{m+j}(x); \quad m \geq 0. \qquad (6.2)$$

The action of the projection operators $\prod_{[n,k]}$ on both sides of (6.2); with $n = r$; gives us, by the virtue of the theorem III-1 in [2], a $(2n + 1)$-term recurrence relation satisfied by the family $\{\prod_{[n,k]} P_m(x)\}_{m=0}^{\infty}$, that is

$$x^n \prod_{[n,k]} P_m(x) = \sum_{j=\sup(-m,-r)}^{r} \alpha_{m\ m+j}^{(n)} \; \prod_{[n,k]} P_{m+j}(x); \quad m \geq 0. \quad (6.3)$$

Let us note that, even though in the beginning of this section we have mentioned "orthogonal polynomials", the result obtained follows only from the validity of the recursion relation (6.1), indeed, analogous results hold for polynomials characterized by recursion relations involving more than three terms and a polynomial in $\Omega_{[n,l]}$; l fixed in $0, 1, \dots, n - 1$; instead of the coefficient x.

We return now to the Laguerre polynomials $\{L_m^{(\alpha)}\}_{m\in\mathbb{N}}$ which satisfy the recurrence relation (see, for instance, [11], p.188 Eq.(8)):

$$xL_m^{(\alpha)}(x) = -(m - 1)L_{m+1}(x) + (2m + \alpha + 1)L_m^{(\alpha)}(x) - (m - \alpha)L_{m-1}^{(\alpha)}(x),$$

with $L_{-1}^{(\alpha)} = 0$ and $L_0^{(\alpha)} = 1$. From (6.3) we deduce, for instance, that

(i) the polynomials $\{L_m^{(\alpha,2,k)}\}_{m\in\mathbb{N}}$ satisfy the five-term recurrence relation:

$$x^2 L_m^{(\alpha,2,k)}(x) = (m+1)(m+2)L_{m+2}^{(\alpha,2,k)}(x)-$$

$$-2(m+1)(2m+2+\alpha)L_{m+1}^{(\alpha,2,k)}(x)+$$

$$+\left[(m+1)(m+1+\alpha)+(2m+\alpha+1)^2+m(m+\alpha)\right]L_m^{(\alpha,2,k)}(x)-$$

$$-2(m+\alpha)(2m+\alpha)L_{m-1}^{(\alpha,2,k)}(x)+$$

$$+(m+\alpha)(m+\alpha-1)L_{m-2}^{(\alpha,2,k)}(x); \quad m \geq 0,$$

with $L_0^{(\alpha,2,0)}(x) = 1$; $L_1^{(\alpha,2,0)}(x) = \alpha + 1$; $L_0^{(\alpha,2,1)}(x) = 0$; $L_1^{(\alpha,2,1)}(x) = -x$; and $L_r^{(\alpha,2,k)}(x) = 0$ for $k = 0,1$ and $r = -2,-1$.

(ii) the polynomials $\{L_m^{(0,3,k)}\}_{m\in\mathbb{N}}$ satisfy the seven-term recurrence relation:

$$x^3 L_m^{(0,3,k)}(x) = -(m+1)(m+2)(m+3)L_{m+3}^{(0,3,k)}(x)+$$

$$+3(m+2)(m+1)(2m+3)L_{m+2}^{(0,3,k)}(x)-$$

$$-3(m+1)\left[5(m+1)^2+1\right]L_{m+1}^{(0,3,k)}(x)+$$

$$+2\left(10m^3+15m^2+11m+3\right)L_m^{(0,3,k)}(x)-$$

$$-3m(5m^2+1)L_{m-1}^{(0,3,k)}(x)+$$

$$+3m(m-1)(2m-1)L_{m-2}^{(0,3,k)}(x)-$$

$$-m(m-1)(m-2)L_{m-3}^{(0,3,k)}(x),$$

with $L_r^{(\alpha,3,k)}(x) = 0$ for $k = 0,1,2$ and $r = -3,-2,-1$, and

$$L_0^{(0,3,0)}(x) = 1; \quad L_1^{(0,3,0)}(x) = 1; \quad L_2^{(0,3,0)}(x) = 1,$$

$$L_0^{(0,3,1)}(x) = 0; \quad L_1^{(0,3,1)}(x) = -x; \quad L_2^{(0,3,1)}(x) = -2x,$$

$$L_0^{(0,3,2)}(x) = 0; \quad L_1^{(0,3,2)}(x) = 0; \quad L_2^{(0,3,2)}(x) = 1/2x^2.$$

7   Differential Equation

Recall that the Laguerre polynomials satisfy the following differential equa-

tion (see, for instance, [1], p.781 Eq.(22.6.15)):

$$\mathcal{L}L_m^{(\alpha)}(x) \equiv \left(xD^2 + (\alpha - 1)D + (m - xD)\right) L_m^{(\alpha)}(x) = 0; \quad D \equiv \frac{d}{dx}.$$

Using the decomposition of the differential operator $\mathcal{L}$ we establish in [4] the following $2n$-th order differential equation satisfied by the components $L_m^{(\alpha,n,k)}$, that is,

$$\left((xD + \alpha - 1)_n D^n - (xD - m)_n\right) L_m^{(\alpha,n,k)}(x) = 0, \qquad (7.1)$$

which, for $n = 2$, reduces to

$$(x^2 D^4 + 2(\alpha + 2)xD^3 + \left((\alpha + 1)(\alpha + 2) + x^2\right) D^2 +$$

$$+2(m - 1)xD - m(m - 1))L_m^{(\alpha,2,k)}(x) = 0.$$

8 Finite Sums

The Laguerre polynomials satisfy a large number of useful summation and multiplication formulas including (cf.[11], p.192 Eq.(41))

$$L_m^{(\alpha+\beta+1)}(x + y) = \sum_{r=0}^{m} L_r^{(\alpha)}(x) L_{m-r}^{(\beta)}(y); \qquad (8.1)$$

in particular, for $\beta = y = 0$, we have

$$L_m^{(\alpha+1)}(x) = \sum_{r=0}^{m} L_r^{(\alpha)}(x); \qquad (8.2)$$

$$L_{m_1}^{(\alpha_1)}(x_1 t) \cdots L_{m_r}^{(\alpha_r)}(x_r t) = \sum_{s=0}^{m_1+m_2+\cdots+m_r} \gamma_s(x_1, \ldots, x_r) L_s^{(\alpha)}(t), \qquad (8.3)$$

(cf.[10], p.156 Eq.(5)), where $\gamma_s(x_1, ..., x_r)$ is a certain hypergeometric polynomial in r variables.

$$L_m^{(\alpha)}(xy) = \sum_{l=0}^{m} \begin{pmatrix} m + \alpha \\ m - l \end{pmatrix} y^l L_l^{\beta}(x) \ _2F_1\left(\begin{array}{cc} -m + l & \beta + l + 1 \\ & \alpha + l + 1 \end{array} ; y \right) \qquad (8.4)$$

((cf.[21], p.68 or [23], p.663 Eq.(4.4)).

From these formulas corresponding ones for $L_m^{(\alpha,n,k)}$ can be obtained by simple manipulations, indeed, if we multiply the variables involved in (8.1) and (8.3) by z and then we apply the projection operators $\prod_{[n,k]}$ to the two

members; viewed as functions of the variable z; of each identity obtained, we state; by the virtue of the property (II-4) in [2]; the following formulas when $z = 1$:

$$L_m^{(\alpha+\beta+1,n,k)}(x+y) = \sum_{r=0}^{m} \sum_{p+q\equiv k(n)} L_r^{(\alpha,n,p)}(x) L_{m-r}^{(\beta,n,q)}(y); \qquad (8.5)$$

$$L_m^{(\alpha+1,n,k)}(x) = \sum_{r=0}^{m} L_r^{(\alpha,n,k)}(x); \qquad (8.6)$$

$$\sum_{k_1+k_2+\cdots k_r\equiv k(n)} L_{m_1}^{(\alpha_1,n,k_1)}(x_1 t) \cdots L_{m_r}^{(\alpha_r,n,k_r)}(x_r t) =$$

$$(8.7)$$

$$= \sum_{s=0}^{m_1+m_2+\cdots+m_r} \gamma_s(x_1,\ldots,x_r) L_s^{(\alpha,n,k)}(t).$$

Also, a mere mechanical application of the projection operators $\prod_{[n,k]}$ on both sides of (8.4); viewed as functions of the variable x; gives us

$$L_m^{(\alpha,n,k)}(xy) = \sum_{l=0}^{m} \binom{m+\alpha}{m-l} y^l L_l^{\beta,n,k}(x) \; {}_2F_1\left(\begin{array}{cc} -m+l, & \beta+l+1 \\ & \alpha+l+1 \end{array}; y\right).$$

$$(8.8)$$

Yet, another finite sum can be deduced from (8.1) by the virtue of the corollary III-3 in [2], that is,

$$_n\tau_y L_m^{(\alpha+\beta+1,n,k)}(x) = \sum_{r=0}^{m} L_r^{(\alpha,n,k)}(x) L_{m-r}^{(\beta,n,0)}(y), \qquad (8.9)$$

where $_n\tau_y; y \in \mathbb{C}$; is the n-translation operator defined by (cf [2], section III):

$$_n\tau_y f(x) = \frac{1}{n} \sum_{k=0}^{n-1} f\left(x + \omega_n^k y\right)$$

Next, we express for $L_m^{(\alpha)}$ two identities which involve $L_m^{(\alpha)}(\omega_n^l z); \; l = 0, 1, \ldots, n-1$; by the components $L_m^{(\alpha,n,k)}(z); \; k = 0, 1, \ldots, n-1$; The first one can be deduced from the Parseval formula (cf.[2], Eq.(V-2)):

$$\sum_{l=0}^{n-1} \left| L_m^{(\alpha)}\left(\omega_n^l x\right) \right|^2 = n \sum_{l=0}^{n-1} \left| L_m^{(\alpha,n,k)}(x) \right|^2 \qquad (8.10)$$

and the second one is a consequence of the n-th order circulant determinant (VI-3) in [2],

$$
\prod_{l=0}^{n-1} L_m^{(\alpha)}\left(\omega_n^l x\right) =
\begin{vmatrix}
L_m^{(\alpha,n,0)}(x) & L_m^{(\alpha,n,n-1)}(x) & \dots & L_m^{(\alpha,n,1)}(x) \\
L_m^{(\alpha,n,1)}(x) & L_m^{(\alpha,n,0)}(x) & \dots & L_m^{(\alpha,n,2)}(x) \\
\vdots & \vdots & \ddots & \vdots \\
L_m^{(\alpha,n,n-1)}(x) & L_m^{(\alpha,n,n-2)}(x) & \dots & L_m^{(\alpha,n,0)}(x)
\end{vmatrix}.
$$

(8.11)

The coefficients of this n-th order determinant are polynomials which degrees are less than or equal to m, so, following Parodi (cf.[17], p.158 Eq.(VI-8)), we can express (8.11) under the form

$$
\prod_{l=0}^{n-1} L_m^{(\alpha)}\left(\omega_n^l x\right) = \det\left(M - x I_{nm}\right),
$$

where M is a $nm \times nm$-matrix which coefficients are complex numbers deduced from the coefficients of the polynomial $L_m^{(\alpha)}$ and I_r designates the $r \times r$ identity matrix.

Notice, also, that the n-th order determinant in (8.11) can be expressed by another way using (8.3).

9   Converse of the First Component

In this section, we need especially the following definitions:

Definition 1: (cf.Douak-Maroni [9], p.83)
A polynomial sequence $\{P_m\}_{m\in\mathbb{N}}$ is called n-symmetric if it fulfils for all $m \in \mathbb{N}$,

$$
\begin{aligned}
&\cdot \deg P_m = m, \\
&\cdot P_m(\omega_n x) = \omega_n^m P_m(x).
\end{aligned}
$$

Definition 2: (cf.Boas-Buck [6], p.18)
A polynomial sequence $\{P_m\}_{m\in\mathbb{N}}$ has a Brenke representation if it is generated by the formal relation:

$$
A(t)C(xt) = \sum_{m=0}^{\infty} P_m(x)t^m;
$$

(9.1)

where

$$\begin{cases} A(t) = \sum_{m=0}^{\infty} a_m t^m, & a_0 \neq 0; \\[2mm] C(t) = \sum_{m=0}^{\infty} c_m t^m, & noc_m = 0. \end{cases}$$

The choice $C(t) = e^t$ gives Appell polynomials and

$$C(t) = {}_0F_l \left(\begin{array}{c} - \\ \beta_1, \ldots, \beta_l \end{array} ; \sigma x t \right);$$

σ a nonzero constant gives Skeff A-type l polynomials (cf. Huff-Raiville [13], p.297).

Consider the polynomials $Q_m^{(\alpha,n)}$; $\quad m \in \mathbb{N}$, defined by

$$Q_m^{(\alpha,n)}(z) = \frac{1}{(\alpha+1)_m} z^m L_m^{(\alpha,n,0)} \left(\frac{1}{z} \right).$$

By changing in (5.6) t by ty and x by x/y, we derive; when $x = 1$; the generating function for the polynomials $Q_m^{(\alpha,n)}$:

$$\sum_{m=0}^{\infty} Q_m^{(\alpha,n)}(z) t^m = e^{zt} (j_\alpha)_{[2n,0]} \left(2\sqrt{t} \right),$$

that means that the system $\{Q_m^{(\alpha,n)}\}_{m \in \mathbb{N}}$ has an Appell representation and since the function $A(t) = (j_\alpha)_{[2n,0]}(2\sqrt{t})$ belongs to $\Omega_{[n,0]}$ (cf.[2], Eq.(II-4)) the use of the proposition III-5 in [5] leads us to state the following

Corollary :. -

(i) The polynomial sequence $\{Q_m^{(\alpha,n)}(z)\}$ $m \in \mathbb{N}$ is n-symmetric.

(ii) The polynomial sequence $\{Q_m^{(\alpha,n)}(z^{1/2})\}$ $m \in \mathbb{N}$ is of Sheffer A-type $(n-1)$.

(iii) $Q_{nm+k}^{(\alpha,n)}(x) = x^k P_m^{(k)}(x^n)$, where the polynomial sequence $\{P_m^{(k)}(x^n)\} m \in \mathbb{N}$ is generated by the relation:

$$\sum_{m=0}^{\infty} P_m^{(k)}(z) t^m = \widetilde{h_{n,k}}(xt) \widetilde{(j_\alpha)}_{[2n,0]} \left(2\sqrt{t} \right),$$

or, equivalently,

$$\sum_{m=0}^{\infty} P_m^{(k)}(z) t^m = \frac{1}{k!} \; {}_0F_{n-1} \left(\begin{array}{c} - \\ \Delta^*(n,k+1) \end{array} ; \frac{xt}{n^n} \right) \times$$

$$\times \; {}_0F_{2n-1} \left(\begin{array}{c} - \\ \Delta^*(n,1)\Delta(n,\alpha+1) \end{array} ; \frac{(-1)^t}{n^{2n}} \cdot t \right).$$

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