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ON THE PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURE ON FINDING EQUISTRONG HOLES IN A SQUARE

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Abstract. In the present work we consider one inverse problem of statics in the linear theory of elastic mixture for a square which is weakened by four unknown equal holes, whose boundaries are free from external forces, and the sides of the square are under the action of absolutely rigid punches of rectilinear base.

Unknown boundaries of the holes are found under the condition that tangential normal stress takes on them one and the same constant value.

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1^0 The homogeneous equation of statics of the linear theory of elastic mixture in the complex form is written as [1]

$$\frac{\partial^2 U}{\partial z \partial \bar{z}} + K\frac{\partial^2 \bar{U}}{\partial \bar{z}^2} = 0, \qquad U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix}, \tag{1}$$

where $u_p, \quad p = \overline{1,4}$ are components of the displacement vector,

$$z = x_1 + ix_2, \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad K = -\frac{1}{2}em^{-1},$$

$$e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \quad m^{-1} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}^{-1}. \quad m_k = e_k + \frac{1}{2}e_{3+k},$$

the $e_q, \;\; q = \overline{1,6}$ are expressed in terms of the elastic mixture [1].

In [1] M. Basheleishvili obtained the representations:

$$2\mu U = 2\mu(u_1 + iu_2, u_3 + iu_4)^T = A\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}, \tag{2}$$

$$TU = \begin{pmatrix} (TU)_2 - i(TU)_1 \\ (TU)_4 - i(TU)_3 \end{pmatrix} = \begin{pmatrix} r'_{12}n_1 + r'_{22}n_2 - i(r'_{11}n_1 + r'_{21}n_2) \\ r''_{12}n_1 + r''_{22}n_2 - i(r''_{11}n_1 + r''_{21}n_2) \end{pmatrix}$$

$$= \frac{\partial}{\partial s(x)}((A - 2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}), \tag{3}$$

where $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions, $(TU)_p$, $p = \overline{1,4}$, are the components of stress vector,

$$\frac{\partial}{\partial s(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}, \qquad n = (n_1, n_2)^T \qquad is \quad unit \quad vector$$

$$A = 2\mu m, \qquad \mu = \begin{bmatrix} \mu_1 \mu_3 \\ \mu_3 \mu_2 \end{bmatrix}, \qquad B = \mu e, \qquad E = \begin{bmatrix} 10 \\ 01 \end{bmatrix},$$ μ_1, μ_2 and μ_3 are elastic constants [1].

Let us now consider the vectors:

$$U_n = (u_1 n_1 + u_2 n_2; u_3 n_1 + u_4 n_2)^T, \quad U_s = (u_2 n_1 - u_1 n_2; u_4 n_1 - u_3 n_2)^T,$$

$$\sigma_n = \begin{pmatrix} (TU)_1 n_1 + (TU)_2 n_2 \\ (TU)_3 n_1 + (TU)_4 n_2 \end{pmatrix}, \sigma_s = \begin{pmatrix} (TU)_2 n_1 - (TU)_1 n_2 \\ (TU)_4 n_1 - (TU)_3 n_2 \end{pmatrix},$$

$$\sigma_t = \begin{pmatrix} [r'_{21} n_1 - r'_{11} n_2; r'_{22} n_1 - r'_{12} n_2]^T s \\ \left[r''_{21} n_1 - r''_{11} n_2; r''_{22} n_1 - r''_{12} n_2\right]^T s \end{pmatrix}. \tag{4}$$

Here $n = (n_1, n_2)^T = (\cos\alpha \quad sin\alpha)^T$, $\quad s = (-n_2, n_1)^T = (-\sin\alpha, \cos\alpha))^T$, and $\alpha(t)$ is an angle between the outer normal to the contour L of the point t and ox_1 axis. Let us call the vector (4) tangential normal stress vector in the linear theory of elastic mixture.

Elementary calculations result in [4]

$$\sigma_n + \sigma_t = (2E - A - B) \; Re\varphi'(t), \quad t \in L, \tag{5}$$

$$\sigma_n + 2\mu \left(\frac{\partial U_s}{\partial s} + \frac{U_n}{\rho_0} \right) + i \left[\sigma_s - 2\mu \left(\frac{\partial U_n}{\partial s} - \frac{U_s}{\rho_0} \right) \right] = 2\varphi'(t) \;\; t \in L, \tag{6}$$

$$[(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}]_L = -i \int_L e^{i\alpha}(\sigma_n + i\sigma_s) ds, \tag{7}$$

where $\dfrac{1}{\varrho_0}$ is the curvature of the curve L at the point t.

2^0 in the work, in the case of the linear theory of elastic mixtures we study the problem analogous to that solved in [2]. For the solution of the problem the use will be made of the generalized Kolosov-Muskhelishvili formula and the method developed in [2] and [4].

Let an isotropic elastic mixture occupy on the plane $z = x_1 + ix_2$ a multiply connected domain G, which is square with vertices lying on the coordinate axes weakened by four unknown equal holes. The holes are intersected by the square diagonals and are symmetric both with respect to these diagonals and to the straight lines connecting middle points of the opposite square sides. The boundaries of the holes are assumed to be free from external loads, the square sides are under the action of absolutely rigid punches of rectilinear base, and concentrated forces $P = (p_1, p_2)^T$ are applied to the middle points of the punches.

Assume that the vector σ_s is equal to zero on the entire boundary G, also $\sigma_n = 0$ on the unknown part of the boundary G. Further note that the vector U_n takes on sides square constant value. Suppose also that the surfaces of the bodies are assumed to be absolutely smooth, and hence the frictional force will be neglected.

The problem is formulated as follows: Find unknown holes and stressed state of the square under the condition that the tangential normal stress σ_t at the hole boundaries takes constant value. Let $\sigma_t = -K^0, \quad K^0 = (K_1^0, K_2^0) = const.$

Since the problem is axially symmetric, we consider a curvilinear pentagon $A_1A_2A_3$ A_4A_5 (Figure 1).

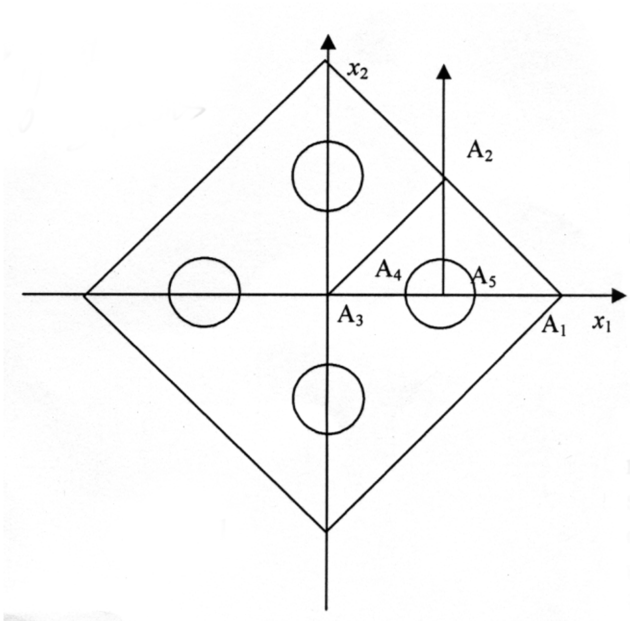

Figure 1:

Introduce the notation $A_kA_{k+1} = \Gamma_k, \quad k = 1, 2, 3, \quad \Gamma_4 = A_5A_1, \quad \Gamma = \bigcup_{k=1}^{4} \Gamma_k$. Let us denote the arc A_4A_5 by Γ_5 and the domain occupied by the curvilinear pentagon by D. Let $2d^0$ be the square diagonal.

On the basis of analogous Kolosov-Muskhelishvilis formulas (5)-(7) our problem is reduced to finding two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in D by the boundary condtions:

$$Re\varphi'(t) = \frac{1}{2}(A + B - 2E)^{-1}K^0, \quad t \in \Gamma_5, \quad Im\varphi'(t) = 0, \quad t \in \Gamma, \tag{8}$$

$$(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)} = q^0, \ \ t \in \Gamma_5, \quad q_0 = const, \tag{9}$$

$$Ree^{-i\alpha(t)}[(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}] = C(t), \quad t \in \Gamma, \tag{10}$$

where $\alpha(t)$ is the size of the angle made by the normal and the ox_1 axis,

$$C(t) = \int_{A_1}^{t} \sigma_n(t_0)\sin(\alpha(t_0) - \alpha(t))ds_0, \quad t \in \Gamma, \quad If \qquad t \in \Gamma_j,$$

then

$$C(t) = 0, \quad t \in \Gamma_1 \cup \Gamma_3 \cup \Gamma_4, \qquad C(t) = \frac{1}{2}P, \quad t \in \Gamma_2.$$

The conditions (8) are the vector-form of the Keldysh-Sedov problem for the domain D. It is proved that

$$\varphi(z) = \frac{1}{2}(A + B - 2E)^{-1}K^0 z + (A - 2E)^{-1}l^0,$$

$$z \in D,\ l^0 = const,\ Iml^0 = 0. \tag{11}$$

$$If \quad t \in \Gamma_k, \quad k = \overline{1,4}, \quad then \quad Re(e^{-i\alpha_k}t) = Re(e^{-i\alpha_k}A_k), \quad t \in \Gamma_k, \quad k = \overline{1,4},$$

$$\alpha_1 = \frac{\pi}{4}, \quad \alpha_2 = \frac{3}{4}\pi, \quad \alpha_3 = \alpha_4 = \frac{3}{2}\pi.$$

Taking into accound equality (11), we can rewrite the boundary conditions (9) and (10) as follows:

$$\frac{1}{2}K^0 t + 2\mu\overline{\psi(t)} = q^0 - l^0, \quad t \in \Gamma_5,$$

$$2\mu Re(e^{-i\alpha(t)}\overline{\psi(t)}) = -\begin{cases} Re e^{-i\alpha(t)}\left(\frac{1}{2}K^0 t + l^0\right), t \in \Gamma_1 \cup \Gamma_3 \cup \Gamma_4, \\ Re e^{-i\alpha(t)}\left(\frac{1}{2}K^0 t + l^0\right) - \frac{1}{2}P, t \in \Gamma_2. \end{cases} \tag{12}$$

Further note that

$$Re(e^{-i\alpha(t)}t) = \frac{\sqrt{2}}{2}d^0, t \in \Gamma_1, Re(e^{-i\alpha(t)}t) = 0, t \in \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \tag{13}$$

Let the function $z = w(\zeta), \zeta = \xi_1 + i\xi_2$ map conformaly domain D onto semi-circle $|\zeta| < 1, Im\zeta > 0.$ In addition, we may assume that the arc $A_4A_5$is mapped onto the diameter $(-1,1); A_4 \to \beta_4 = -1, A_5 \to \beta_5 = 1, A_2 \to \beta_2 = i.$ We map two points A_1 and A_3 onto the unknown points β_1 and β_3.

If we introduce

$$W(\zeta) = \begin{cases} \frac{1}{2}K^0 w(\zeta), |\zeta| < 1, Im\zeta > 0, \\ -2\mu\overline{\psi_0(\overline{\zeta})} + q^0 - l^0, |\zeta| < 1, I_m\zeta < 0, \psi_0(\zeta) = \psi(w(\zeta)), \end{cases} \tag{14}$$

then the boundary value problems (12)-(13) (see [2]) are reduced to the Riemann-Hilbert problem for the circle $|\zeta| < 1$

$$Re\left((e^{-i\alpha(\sigma)}W(\sigma)) = f(\sigma), \sigma \in \gamma, Re(e^{-i\alpha(\sigma)}W(\sigma))\right) = f^0(\sigma), \sigma \in \gamma^0, \tag{15}$$

where $\gamma = \bigcup\limits_{k=1}^{4} \gamma_k, \gamma_k = \omega^{-1}(\Gamma_k), k = \overline{1,4}$ and γ^0 is the mirror image of γ with respect to the diameter (-1,1).

A solution of the problem (15) can be represented in the form [3] and [2]

$$W(\zeta) = \frac{\aleph(\zeta)}{2\pi i} \int_{\gamma \bigcup \gamma^0} \frac{\zeta + \sigma}{\sigma - \zeta} \frac{F(\sigma)}{\sigma \aleph(\sigma)} d\sigma, F(\sigma) = \begin{cases} f(\sigma), \sigma \in \gamma, \\ f^0(\sigma), \sigma \in \gamma^0. \end{cases}$$

$$\aleph(\zeta) = \exp\left(\frac{1}{4\pi i} \int\limits_{\gamma \bigcup \gamma^0} \frac{\zeta + \sigma}{\sigma - \zeta} \frac{2i\alpha(\sigma) d\sigma}{\sigma} \right) = \frac{\aleph_1(\zeta)}{\sqrt{\aleph_1(0)}},$$

$$\aleph_1(\zeta) = \sqrt[4]{\frac{\zeta - \beta_2}{\zeta - \beta_1} \left(\frac{\zeta - \beta_3}{\zeta - \beta_2}\right)^3 \left(\frac{\zeta - \overline{\beta_3}}{\zeta - \beta_3}\right)^2 \left(\frac{\zeta - \overline{\beta_2}}{\zeta - \overline{\beta_3}}\right)^3 \frac{\zeta - \overline{\beta_1}}{\zeta - \overline{\beta_2}} \left(\frac{\zeta - \beta_1}{\zeta - \overline{\beta_1}}\right)^2}.$$

Having known $W(\zeta)$ we can define $\psi_0(\zeta)$ and $\omega(\zeta)$ by (14) and the stressed state of the body and the boundaries of unknown holes.

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