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SOLUTION OF THE BASIC PLANE BOUNDARY VALUE PROBLEMS OF STATICS OF THE ELASTIC MIXTURE FOR A MULTIPLY CONNECTED DOMAIN BY THE METHOD OF D. SHERMAN

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Abstract. In the present work we consider the basic plane boundary value problems of statics of the linear theory of elastic mixture for a multiply connected finite domain, when on the boundary a displacement vector (the first problem) and a stress vector (the second problem) are given.

For the solution of the problem we use the generalized Kolosov-Muskhelishvili formulas and the method of D. Sherman.

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1. Introduction

The construction and the intensive investigation of the mathematical models of elastic mixtures arise by the wide use of composites into practice. The diffusion and shift models of the linear theory of elastic mixtures are presented by several authors.

In [1,3.4] for a simply connected finite and infinite domain the basic plane boundary value problems of statics of the elastic mixture theory are considered when on the boundary a displacement vector (the first problem), a stress vector (the second problem); differences of partial displacements and the sum of stress vector components (the third problem) are given.

In [1] two-dimensional boundary value problems of statics are investigated by potential method and the theory of singular integral equations.

In [3] by applying the general Kolosov-Muskhelishvili representations from ([2]) these problems are splitted and reduced to the first and the second boundary value problem for an elliptic equation which structurally coincides with an equation of statics of an isotropic elastic body.

In [4] using potentials with complex densities the solutions of basic plane boundary value problems of statics are reduced to solution of Fredholm linear integral equation of second kind.

In [5] the basic mixed boundary value problem of equation of statisc of the elastic mixture theory is considered in a simply connected domain when the displacement vector is given on one part of the boundary and the stress vector on the remaing part.

In [7] three - dimensional boundary value problems of two isotropic elastic medea are investigated by means of the potential method. The uniqueness and existence theorems for the statics, steady oscillations and dynamical problems are proved.

In the present work in the case of the plane theory of elastic mixture for a multiply connected finite domain we study the problems the variant of which in the case of the plane theory of elasicity has been solved by N. Muskhelishvili, owing to the method of D. Sherman [6, §102]

For the solution of the problem the use will be made of the generalized Kolosov-Muskhelishvuli's formula [2,4] and the method D. Sherman developed in [6; §102].

2. Some auxiliary formulas and operators

The homogeneous equation of statics of the theory of elastic mixture in the complex form is written as [4]

$$\frac{\partial^2 U}{\partial z \partial \bar{z}} + K \frac{\partial^2 \bar{U}}{\partial \bar{z}^2} = 0 \tag{2.1}$$

where $U = (u_1 + iu_2, u_3 + iu_4)^T$, $u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$ are partial displacements, $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$, $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$,

$$K = -\tfrac{1}{2} l m^{-1},\ l = \begin{bmatrix} l_4 & l_5 \\ l_5 & l_6 \end{bmatrix},\ m^{-1} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}^{-1},$$

$m_k = l_k + \frac{1}{2}\, l_{3+k}$, $k = 1, 2, 3$, $l_1 = a_2/d_2$, $l_2 = -c/d_2$, $l_3 = a_1/d_2$, $a_1 = \mu_1 - \lambda_5$, $a_2 = \mu_2 - \lambda_5$, $c = \mu_3 + \lambda_5$, $d_2 = a_1 a_2 - c^2$, $l_1 + l_4 = b/d_1$, $l_2 + l_5 = -c_0/d_1$, $l_3 + l_6 = a/d_1$, $a = a_1 + b_1$, $b = a_2 + b_2$, $c_0 = c + d$, $b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2\rho_2/\rho$, $b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2\rho_1/\rho$, $d = \mu_3 + \lambda_3 - \lambda_5 - \alpha_2\rho_1/\rho \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2\rho_2/\rho$, $\alpha_2 = \lambda_3 - \lambda_4$, $\rho = \rho_1 + \rho_2$, $d_1 = ab - c^2$.

ρ_1 and ρ_2 appearing in (2.2) are the partial densities, and $\mu_1, \mu_2, \mu_3, \lambda_p$, $p = \overline{1,5}$ are real constants characterizing physical properties of the elastic mixture and satisfying certain inequalities [1] and [7].

Let D^+ be a bounded two-dimensional domain (surrounded by the curve S) and let D^- be the complement of $\bar{D}^+ = D^+ US$. We assume that $S \in C^{k+\beta}$, $k = 1, 2$, $0 < \beta \leq 1$.

A vector $u = (u', u'')^T = (u_1, u_2, u_3, u_4)^T$ is said to be regular in $D^+[D^-]$ if $u_k \in C^2(D^+) \bigcap C^1(\bar{D^+})$ $[u_k \in C^2(D^-) \bigcap C^1(\bar{D^-})]$ and the second order derivatives of u_k are summable in $D^+[D^-]$, in the case of the domain D^- we assume, in addition the following conditions at infinity

$$u_k(x) = 0\,(1), \qquad |x|^2 \frac{\partial u_k}{\partial x_j} = 0(1), \qquad j = 1, 2; \qquad k = \overline{1,4},$$

to be fulfiled with $|x|^2 = x_1^2 + x_2^2$.

In [2] M. Basheleishvili obtained the following representations

$$U = (U_1, U_2)^T = (u_1 + iu_2, u_3 + iu_4)^T = m\varphi(z) + \frac{1}{2}\, l\ \ z\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{2.3}$$

$$\begin{aligned} TU &= [(TU)_1, (TU)_2]^T = [(Tu)_2 - i(Tu)_1, (Tu)_4 - i(Tu)_3]^T \\ &= \frac{\partial}{\partial s(x)}[(A - 2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}], \end{aligned} \tag{2.4}$$

where $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)$ are arbitrary analytic vector-functions,

$$A=\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}=2\mu m,\ \mu=\begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix},\ m=\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix},$$

$$B=\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}=\mu l,\ E=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

are known matrices and (see [5])

$$A_1+A_3-2=B_1+B_3,\quad A_2+A_4-2=B_2+B_4, \tag{2.5}$$

$\det\ m>0, \det\ \mu>0,\ \det(A-2E)>0.$

$\dfrac{\partial}{\partial S(x)}=n_1\dfrac{\partial}{\partial x_2}-n_2\dfrac{\partial}{\partial x_1},\quad n=(n_1,n_2)^T$ is a unit vector of the outer normal

$(Tu)_p,\ p=\overline{1,4}$ are the components of stresses [2]

$(Tu)_1=r'_{11}n_1+r'_{21}n_2,\quad (Tu)_2=r'_{12}n_1+r'_{22}n_2,$

$(Tu)_3=r''_{11}n_1+r''_{21}n_2,\quad (Tu)_4=r''_{12}n_1+r''_{22}n_2,$

$$\tau^{(1)}=\begin{pmatrix} r'_{11} \\ r''_{11} \end{pmatrix}=\begin{bmatrix} a & c_0 \\ c_0 & b \end{bmatrix}\begin{pmatrix} \theta' \\ \theta'' \end{pmatrix}-2\mu\frac{\partial}{\partial x_2}\begin{pmatrix} u_2 \\ u_4 \end{pmatrix},$$

$$\tau^{(2)}=\begin{pmatrix} r'_{22} \\ r''_{22} \end{pmatrix}=\begin{bmatrix} a & c_0 \\ c_0 & b \end{bmatrix}\begin{pmatrix} \theta' \\ \theta'' \end{pmatrix}-2\mu\frac{\partial}{\partial x_1}\begin{pmatrix} u_1 \\ u_3 \end{pmatrix},$$

$$\eta^{(1)}=\begin{pmatrix} \eta'_{21} \\ \eta''_{21} \end{pmatrix}=-\begin{bmatrix} a_1 & c \\ c & a_2 \end{bmatrix}\begin{pmatrix} \omega' \\ \omega'' \end{pmatrix}+2\mu\frac{\partial}{\partial x_1}\begin{pmatrix} u_2 \\ u_4 \end{pmatrix},$$

$$\eta^{(2)}=\begin{pmatrix} r'_{12} \\ r''_{12} \end{pmatrix}=\begin{bmatrix} a_1 & c \\ c & a_2 \end{bmatrix}\begin{pmatrix} \omega' \\ \omega'' \end{pmatrix}+2\mu\frac{\partial}{\partial x_2}\begin{pmatrix} u_1 \\ u_3 \end{pmatrix}. \tag{2.6}$$

$\theta'=divu',\quad \theta''=divu'',\quad \omega'=rotu',\quad \omega''=rotu''.$

By virtue of (2.2) and (2.6) we obtain lengthy but elementary calculations.

$$\tau=\tau^{(1)}+\tau^{(2)}=2(2E-A-B)Re\ \ \varphi'(z),$$

$$\tau^{(1)}-\tau^{(2)}-i\eta=2[B\overline{z}\varphi''(z)+2\mu\psi'(z)],\quad \eta=\eta_1+\eta_2, \tag{2.7}$$

$\det(2E-A-B)>0$ (see [2]).

Formulas (2.3), (2.4) and (2.7) are analogous to the Kolosov-Muskhelishvili's formulas for the linear theory of elastic mixture.

Also note that

$$X + iY = i[(A-2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}]_S \tag{2.8}$$

is the principal vector of stresses applied on S.

For our purpose let us rewrite formulas (2.4) in a more convenient form. Namely, for the stress vector we have

$$(A-2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)} = F + \nu, \qquad (2.4)'$$

where $\nu = (\nu_1, \nu_2)^T$ is an arbitrary complex vector,

$$F = (F_1, F_2)^T = \int_{z_0}^{z} TUds,$$

here the integral is taken over any smooth arc within D^+ connecting an arbitrary fixed point z_0 with a variable point z of D^+.

Multiplying $(2.4)'$ by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}\overline{dt}$ and integrating over S. Owing to (2.5) we obtain

$$\begin{pmatrix} B_1 + B_3 \\ B_2 + B_4 \end{pmatrix} \int_S [\varphi(t)\overline{dt} - \overline{\varphi(t)}dt] = \int_S \begin{pmatrix} 1 \\ 1 \end{pmatrix} F(t)\overline{dt}. \tag{2.9}$$

From (2.9) we have $Re \int_S F(t)\overline{dt} = 0$.

Below we will need the following Greens formulas [1] and [4]

$$\int_{D^\pm} E(u,u)dx = \pm\ I_m \int_S U\overline{TU}ds, \tag{2.10}$$

where $E(u,u)$ is the positively defined quadratic form, the equation

$E(u,u) = 0$ admits a solution $u = (u', u'')^T$, $u' = (u_1, u_2)^T = a' + b' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$,

$$u'' = (u_3, u_4)^T = a'' + b' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \tag{2.11}$$

where a' and a'' are arbitrary real constant vectors, and b' is an arbitrary real constant.

Let G^+ be a finite multiply connected domain bounded by the contours $L_1, L_2, L_3, ...,$ L_p, L_{p+1}, the last of which contains all the others, $L_j \in C^{1,\beta}$ $0 < \beta \le 1, j = \overline{1, p+1}$. In this case the boundary of G^+ is $L = \bigcup\limits_{j=1}^{p+1} L_j$; note that the contours $L_j (j \le p)$ are oriented clockwise, while L_{p+1} is oriented counterclockwise. Let $G_j (j = \overline{1,p})$ be a finite two-dimensional domain bounded by the contour L_j, $j = \overline{1,p}$. By G_{p+1} we denote an infinite domain bounded by the contour L_{p+1}. $G' = \bigcup\limits_{j=1}^{p+1} G_j$, and $G^- = R^2 \setminus \bigcup\limits_{j+1}^{p} G_p$.

Note that in a domain G^+ components of the partial displasements and stress vectors are one-valued functions.

Repeating word by word the reasoning developed in [6 §35], owing to formulas (2.7)-(2.8) we obtain that (2.3) represent one-valued vector-function in the domain G^+, when

$$\varphi(z)=\sum_{k=1}^{p}\gamma_k ln(z-z_k)+\varphi^*(z) \tag{2.12}$$

$$\psi(z)=\sum_{k=1}^{p}\gamma_k^{\prime} ln(z-z_k)+\psi^*(z) \tag{2.13}$$

where z_k is an arbitrary point in $G_k, k=\overline{1,p}$

$$\gamma_k=-\frac{X_k+iY_k}{4\pi},\quad \gamma_k^{\prime}=-\frac{m(X_k-iY_k)}{4\pi},$$

$X_k+iY_k=i[(A-2E)\varphi(t)+Bt\overline{\varphi^{\prime}(t)}+2\mu\overline{\psi}(t)]_{L_k}$; $\varphi^*(z)$ and $\psi^*(z)$ are holomorphic vector-functions in G^+.

Finally note that the formula $(2.10)^+$ is valid for domain G^+

$$\int_{G^+}E(u,u)dx=I_m\int_L U\overline{TU}ds$$

$$=I_m\int_L[m\varphi(t)+\frac{1}{2}\ \ lt\overline{\varphi^{\prime}(t)}+\overline{\psi(t)}]d[(A-2E)\overline{\varphi(t)}+B\bar{t}\varphi^{\prime}(t)+2\mu\psi(t)]. \tag{2.14}$$

3. Solution of the first boundary valu problem for the finite multiply connected domain

Let G^+ be a finite multiply connected domain (see section 2). The first boundary value problem is formulated as follows: Find in the domain G^+ a vector $U(x)$ which belongs to the class $C^2(G^+)\bigcap C^{(1,\alpha)}(\overline{G^+})$ is a solution of equation (2.1.) and satisfying the following condition

$$U^+(t_0)=f(t_0)\quad on\quad L,\quad -(I)_f^+\quad problem;$$

where $f\in C^{1,\alpha}(L),\ \ L\in C^{(2,\beta)}, 0<\alpha<\beta\le 1$ is a given complex vector-function.

Using the Green formula (2.14) it is easy to prove.

Theorem 3.1. *The homogeneous problem $(I)_0^+$, has no nontrivial regular solution.*

By virtue of (2.3) it is obvious that the $(I)_f^+$ problem can be reduced to a problem of defining two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in G^+ using the boundary condition

$$U^+(t_0)=m\varphi((t_0)+\frac{1}{2}\quad lt_0\overline{\varphi^{\prime}\ (t_0)}+\overline{\psi(t_0)}=f(t_0), on\ \ L. \tag{3.1}$$

Let us look for analytic vector-functions $\varphi(z)$ and $\psi(z)$ in the form (see (2.12) and (2.13))

$$\varphi(z)=\frac{m^{-1}}{2\pi i}\int_L\frac{g(t)dt}{t-z}+\sum_{j=1}^{p}m^{-1}q_j\ \ ln(z-z_j), \tag{3.2}$$

$$\psi(z)=\frac{1}{2\pi i}\int_L\frac{\overline{g(t)}dt}{t-z}-\frac{K}{2\pi i}\int_L\frac{g(t)\overline{dt}}{t-z}+$$

$$+\frac{K}{2\pi i}\int_L \frac{\bar{t}g(t)dt}{(t-z)^2}+\sum_{j=1}^{p}\overline{q_j}\;\; ln(z-z_j), \tag{3.3}$$

where $z_j = x_{1j} + ix_{2j}$ is a arbitrary point in G_j, $j = \overline{1,p}, z = (x_1 + ix_2) \in G^+$, $g = (g_1, g_2)^T$ is the unknown complex vector to the Hölder class and has the integrable derivative, and $q_j = (q_{j1}, q_{j2})^T$ is an arbitrary constant vector, $(j = \overline{1,p})$.

We tie the unknown constant vector q_j and the unknown vector g by the relation

$$q_j = \int_{L_j} g(t)ds, \qquad j = \overline{1,p}. \tag{3.4}$$

Substituting (3.2) and (3.3) into (2.3.) we have by (3.4) that

$$U(x) = \frac{1}{2\pi i}\int_L g(t)dln\frac{t-z}{\bar{t}-\bar{z}} + \frac{K}{2\pi i}\int_L \overline{g(t)}d\frac{t-z}{\bar{t}-\bar{z}}$$

$$+\sum_{j=1}^{p}\left[2ln|z-z_j|\int_{L_j} g(t)ds - K\frac{z}{\bar{z}-\overline{z_j}}\int_{L_j} g(t)ds\right]. \tag{3.5}$$

Passing to the limit in (3.5) $G^+ \ni z \to t_0 \in L$ and using boundary condition (3.1.) to define the vector g we obtain the following integral equation of Sherman type

$$g(t_0) + \frac{1}{2\pi i}\int_L g(t)dln\frac{t-t_0}{\bar{t}-\overline{t_0}} + \frac{K}{2\pi i}\int_L \overline{g(t)}d\frac{t-t_0}{\bar{t}-\overline{t_0}} + \sum_{j=1}^{p}[2ln|t_0-z_j|$$

$$-K\frac{t_0}{\overline{t_0}-\overline{z_j}}\;\;]\int_{L_j} g(t)ds = f(t_0), \quad t_0 \in L. \tag{3.6}$$

Since $f \in C^{1,\alpha}(L)$, $\quad L \in C^{2,\beta}$ $\quad (0 < \alpha < \beta \leq 1)$, therefore from (3.6) it follows (see [4]) $g \in C^{1,\alpha}(L)$.

Let us show now that equation (3.6) is always solvable. For this it is suficient that the homogeneous equation corresponding to (3.6) has only a trivial solution. Denote the homogeneous equation (which we do not write) by $(3.6.)^0$ and assume that it has a solution different from zero which is denoted by g_0. Compose the complex potentials $\varphi_0(z)$ and $\psi_0(z)$ using (3.2) and (3.3.), where g is replased by g_0. We have

$$U_0(t_0) = m\varphi_0(t_0) + \frac{1}{2}\;\; lt_0\overline{\varphi'(t_0)} + \overline{\psi_0(t_0)} = 0, \qquad t_0 \in L. \tag{3.7}$$

Due to Theorem 3.1. we obtain $\quad u_0(x) = 0, \quad x \in G^+,\;$ hence (see [5])

$$\varphi_0(z) = \nu; \quad \psi_0(z) = -m\bar{\nu}, \tag{3.8}$$

where $\nu = (\nu_1, \nu_2)^T$ is an arbitrary constant vector.

Now note that since vector-functions $\varphi_0(z)$ and $\psi_0(z)$ are one-valued in G^+ therefore by (3.2.) - (3.4.) and (3.8.) we can write

$$\varphi_0(z) = \frac{m^{-1}}{2\pi i}\int_L \frac{g_0 dt}{t-z} = \nu, \qquad z \in G^+,$$

$$\psi_0(z)=\frac{1}{2\pi i}\int_L\frac{\overline{g_0(t)}dt}{t-z}+\frac{K}{2\pi i}\int_L\frac{\bar t g_0'(t)dt}{t-z}=-m\bar\nu,\qquad z\in G^+, \tag{3.9}$$

$$q_j^0=\int_{L_j}g_0(t)ds\qquad j=\overline{1,p}. \tag{3.10}$$

Consider the following vector-functions:

$$i\varphi^*(t)=m^{-1}g_0(t)-\nu;;\qquad i\psi^*(t)=\overline{g_0(t)}+K\bar t g_0'(t)+m\bar\nu. \tag{3.11}$$

By virtue of (3.9.) we obtain

$$\frac{1}{2\pi i}\int_L\frac{\varphi^*(t)dt}{t-z}=0,\qquad \frac{1}{2\pi i}\int_L\frac{\psi^*(t)dt}{t-z}=0,\quad \forall z\in G^+.$$

Hence we conclude, that (see [6, §74]) the vector-functions $\varphi^*(t)$ and $\psi^*(t)$ are the boundary values of the vector functions $\varphi^*(z)$ and $\psi^*(z)$ which are holomorphic in the domains $G_1, G_2, G_3, ..., G_p, G_{p+1}$ and $\varphi^*(\infty)=0$, $\psi^*(\infty)=0$.

After eliminating $g_0(t)$; in (3.11.), we obtain

$$m\overline{\varphi^*(t_0)}+\frac{1}{2}\,l\,\overline{t_0}\varphi^{*\prime}(t_0)+\psi^*(t_0)=-2im\bar\nu,\quad on\quad L_j,\quad j=\overline{1,p+1}.$$

By (2.3.) this condition correspoinds to the first boundary value problem of statics in the elastic mixture theory the domain G_j, $j=\overline{1,p+1}$, when at the body boundary the displacement vector is equal to constants $-2im\bar\nu$.

Using the uniqueness theorem for the domain G_j, $j=\overline{1,p+1}$ (see [4]) we have

$$\varphi^*(z)=c_j,\quad \psi^*(z)=-im\bar\nu-m\bar c_j, in\quad G_j,\ \ j=\overline{1,p+1},$$

where $c_j=(c_{j1},c_{j2})^\tau$, $(j=\overline{1,p+1})$, is an arbitrary constant complex vector.

Since in the domain G_{P+1} $\varphi^*(\infty)=\psi^*(\infty)=0$ therefore $\nu=0$ and $C_{p+1}=0$. Hence $\varphi^*(z)=c_j$, $\psi^*(z)=-m\bar c_j$, in G_j $j=\overline{1,p}$, $\varphi^*(z)=\psi^*(z)=0$ in G_{p+1}.

In that case (3.11) implies

$$m^{-1}g_0(t)=ic_j\quad on\quad L_j,\quad j=\overline{1,p}\quad and\quad g_0(t)=0\quad on\quad L_{P+1}. \tag{3.12}$$

Now on the basis of (3.10) we obtain that every $c_j=0$, hence $g_0(t)=0$.

Consequently the homogeneous equation corresponding to (3.6) has no nontrivial solution. This means that (3.6) has a unique solution. Substituting g in (3.5), we get a solution of the first boundary value problem.

The existence of solution of the first boundary value problem can also be proved when domain G is an infinite multiply-connected domain

4. Solution of the second boundary value problem for the finite multiply connected domain

Let G^+ be a finite multiply connected domain (see section 2). The origin is assumed to lie in the domain G_{P+1}.

The second boundary value problem is investigated with the vector

$TU = ((Tu)_2 - i(Tu)_1, (Tu)_4 - i(Tu)_3)^T$ given on the boundary where $(Tu)_k$, $k = \overline{1,4}$ are the components of stresses (see (2.6).)

Using the Green formula (2.14) it easy to prove.

Theorem 4.1. *The general solution of the second homogeneous boundary value problem, in G^+ is represented by the formula*

$$U = a^0 + i\varepsilon^0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} z,$$

where $z = x_1 + ix_2$, $a^0 = (a_1^0, a_2^0)^T$ is an arbitrary complex constant vector, and ε^0 is an arbitrary constant.

The latter formula expresses a rigid displacement of the body.

It is assumed that the principal vector and the principal moment of external forces are equal to zero on every contour $L_j (j = \overline{1,p})$. Moreover for solvability of the problem we also assume that the principal vector of external forces is equal to zero on L_{P+1}.

By virtue of (2.4) and (2.4)′ it is obvious that the second plane boundary value problem can be reduced to a problem of defining two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in G^+ using the boundary condition

$$(A - 2E)\varphi(t_0) + Bt_0\overline{\varphi'(t_0)} + 2\mu\overline{\psi(t_0)} - \nu_k = F(t_0), \quad \text{on} \quad L_k, \quad k = \overline{1, p+1}, \tag{4.1}$$

where $F = (F_1 F_2)^T \in C^{1,\alpha}(L_k)$, $L_k \in C^{2,\beta}, 0 < \alpha < \beta \leq 1$ is a given vector-function. $\nu_k = (\nu_{k1}, \nu_{k2})^T$, $(k = \overline{1, p+1})$ is a constant vector. Note that the constants $\nu_1, \nu_2, \nu_3, ..., \nu_p, \nu_{p+1}$ are not given in advance and defined while solving the problem, if we fix one of them. Below we will assume that $\nu_{p+1} = 0$.

In (4.1) $\varphi(t_0), \varphi'(t_0)$ and $\psi(t_0)$ denote the boundary values on $L_k, k = \overline{1, p+1}$, of the vector-functions $\varphi(z), \varphi'(z)$ and $\psi(z)$ respectively.

In the sequel we will be assume that

$$Re \int_L \begin{pmatrix} 1 \\ 1 \end{pmatrix} F(t) d\bar{t} = 0. \tag{4.2}$$

Note that (see [6], [4]) condition (4.2) expresses the principal vector and the principal moment of external forces are equal to zero.

The analytic vector-functions $\varphi(z)$ and $\psi(z)$ sought for in the domain G^+ have the form

$$\varphi(z) = \frac{(A - 2E)^{-1}}{2\pi i} \int_L \frac{g(t)dt}{t - z} + \sum_{j=1}^{P} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{M_j}{z - z_j}, \tag{4.3}$$

$$\psi(z) = (2\mu)^{-1} \left[\frac{1}{2\pi i} \int_L \frac{\overline{g(t)}dt}{t - z} + \frac{H}{2\pi i} \int_L \frac{g(t)\overline{dt}}{t - z} - \frac{H}{2\pi i} \int_L \frac{\bar{t}g(t)dt}{(t - z)^2} + \sum_{j=1}^{P} B \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{M_j}{z - z_j} \right] \tag{4.4}$$

where $H = B(A-2E)^{-1}$ is a known matrix, $z_j = x_{1j} + x_{2j}$ is an arbitrary fixed point in $G_j, (j = \overline{1,p}), g = (g_1, g_2)^T$ is a complex unknown vector-function, M_j is a real constant. Then we tie the unknown constant M_j and unknown vector-function g by the relation

$$M_j = i\begin{pmatrix} 1 \\ 1 \end{pmatrix} \int_{L_j} (g(t)\overline{dt} - \overline{g(t)}dt), \quad j = \overline{1,p}. \tag{4.5}$$

Taking into account (4.3) and (4.4) in (4.1) after some calculations for the determination of the vector g we obtain the following equation of Sherman type

$$g(t_0) + \frac{1}{2\pi i}\int_L g(t) d ln \frac{t-t_0}{\overline{t}-\overline{t_0}} - \frac{H}{2\pi i}\int_L \overline{g(t)} d\frac{t-t_0}{\overline{t}-\overline{t_0}}$$
$$+\sum_{j=1}^{p}\left[(A-2E)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{M_j}{t_0 - z_j} + B\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{M_j}{\overline{t_0}-\overline{z_j}} - B\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{M_j t_0}{(\overline{t_0}-\overline{z_j})^2}\right]$$
$$-\nu_k = F(t_0), \quad on \quad L_k, \quad k = \overline{1,p+1}, \tag{4.6}$$

where $\nu_k, \quad k = \overline{1,p}$ are an arbitrary constant vector, $\nu_{p+1} = 0$, and $M_j, \quad j = \overline{1,p}$ are given by (4.5).

We tie the unknown constant vector ν_k and the unknown vector-function g by the relation

$$\nu_k = -\int_{L_k} g(t)ds, \quad k = \overline{1,p}. \tag{4.7}$$

If now in the left-hand side of the second integral equation in (4.6) under the vector ν_k is meant the expression (4.7) then this equation will transform into *a* equation containing no unknown except vector g.

To investigate equation (4.6) it's advisable to consider, instead of (4.6) the equation

$$g(t_0) + \frac{1}{2\pi i}\int_L g(t) d ln \frac{t-t_0}{\overline{t}-\overline{t_0}} - \frac{H}{2\pi i}\int_L \overline{g(t)} d\frac{t-t_0}{\overline{t}-\overline{t_0}}$$
$$+\sum_{j=1}^{p}\left[(A-2E)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{M_j}{t_0 - z_j} + B\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{M_j}{\overline{t_0}-\overline{z_j}} - B\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{M_j t_0}{(\overline{t_0}-\overline{z_j})^2}\right]$$
$$+\frac{1}{4\pi i}\begin{pmatrix} 1 \\ 1 \end{pmatrix} M_{p+1}\left(\frac{1}{t_0} + \frac{1}{\overline{t_0}} - \frac{t}{\overline{t_0^2}}\right) - \nu_k = F(t_0),$$
$$on \quad L_k. \quad k = 1,\overline{p+1}, \tag{4.8}$$

where

$$M_{P+1} = -i\begin{pmatrix} 1 \\ 1 \end{pmatrix}(\varphi'(\xi_0) - \overline{\varphi'(\xi_0)}), \tag{4.9}$$

$\xi_0 = \xi_1^0 + i \ \xi_2^0$ is a fixed point in G^+.

Now note that, by means of analytic vector-functions $\varphi(z)$ and $\psi(z)$ (which are defined by (4.3) and (4.4)) equation (4.8) can be rewritten as

$$(A-2E)\varphi(t_0) + Bt_0\overline{\varphi'(t_0)} + 2\mu\overline{\psi(t_0)} + \frac{1}{4\pi i}\begin{pmatrix} 1 \\ 1 \end{pmatrix} M_{p+1}\left(\frac{1}{t_0} + \frac{1}{\overline{t_0}} - \frac{t}{\overline{t_0^2}}\right)$$

$$-\nu_j = F(t_0) \ \ on \quad L_j, \quad j = \overline{1, p+1}, \tag{4.8$'$}$$

where $\varphi(t_0), \varphi^{'}(t_0)$ *and* $\psi(t_0)$ are boundary values on L_j of the analytic vector-functions $\varphi(t_0), \varphi^{'}(t_0)$ *and* $\psi(t_0)$ respectively.

Multiplying (4.8)$'$ by $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \overline{dt_0}$ and integrating over L. Owing to (2.5) we obtain

$$\begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix} [\varphi(t_0)\overline{dt_0} - \overline{\varphi(t_0)}dt] + \frac{M_{p+1}}{4\pi i} \int_L \left[\frac{\overline{dt_0}}{t_0} + \frac{dt_0}{\overline{t_0}} \right] + M_{p+1}$$

$$= \int_L \begin{pmatrix} 1 \\ 1 \end{pmatrix} F(t_0)\overline{dt_0}.$$

Since M_{p+1} represents a real constant, (see (4.9)), therefore by virtue of (4.2) from the last equalities we find that

$$M_{p+1} = Re \int_L \begin{pmatrix} 1 \\ 1 \end{pmatrix} F(t_0)\overline{dt_0} = 0. \tag{4.10}$$

From (4.10) it follows that the principal vector and the principal moment of eternal forces are equal to zero (see (4.2)), then any solution g of equation (4.8) is simultaneously a solution of the initial equation (4.6).

Let us prove that equation (4.8) is always solvable. To this end it is sufficient to show that the homogeneous equation corresponding to (4.8)has only the trivial solution.Assume the contrary, let g_0 be its solution.Denote the corresponding complex potentials by $\varphi_0(z)$ and $\psi_0(z)$. By virtue of (4.3)-(4.5) and (4.7) we obtain

$$\varphi_0(z) = \frac{(A-2E)^{-1}}{2\pi i} \int_L \frac{g_0(t)dt}{t-z} + \sum_{j=1}^{p} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{M_j^0}{z-z_j}, \tag{4.11}$$

$$\psi_0(z) = \frac{(2\mu)^{-1}}{2\pi i} \int_L \frac{\overline{g_0(t)}dt}{t-z} - \frac{(2\mu)^{-1}H}{2\pi i} \int_L \frac{\bar{t}g_0'(t)dt}{t-z}$$

$$+(2\mu)^{-1} \sum_{j=1}^{p} B \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{M_j^0}{z-z_j}, \tag{4.12}$$

$$\nu_j^0 = -\int_{L_j} g_0(t)ds, \ M_j^0 = i \begin{pmatrix} 1 \\ 1 \end{pmatrix} \int_{L_j} (g_0(t)\overline{dt} - \overline{g_0(t)}dt), \ j = \overline{1,p}. \tag{4.13}$$

Obviously the condition

$$M_{p+1}^0 = -i \begin{pmatrix} 1 \\ 1 \end{pmatrix} (\varphi_0'(\xi_0) - \overline{\varphi_0}(\xi_0)) = 0 \tag{4.14}$$

is fulfilled.

Finally note that, it is easy to see that analytic vector-functions, i.e.complex potentials, $\varphi_0(z)$ and $\psi_0(z)$ satisfy the condition

$$(A-2E)\varphi_0(t_0)+Bt_0\overline{\varphi'(t_0)}+2\mu\overline{\psi_0(t_0)}-\nu_j^0=0, on L_j, j=\overline{1,p+1}, \nu_{p+1}^0=0. \quad (4.15)$$

In that case condition (4.15) corresponds to the boundary condition

$$(TU_0(t_0))^+=0, t_0\in L,$$

where U_0 is obtained from (2.3), if instead of $\varphi(z)$ and $\psi(z)$ we take $\varphi_0(z)$ and $\psi_0(z)$.

Now note that on the basis of uniqueness of Theorem 4.1. we can conclude that solution of the problem (4.15) in the case

$$\nu_j^0=0, j=\overline{1,p+1}, \quad (4.16)$$

is given by

$$U_0=m\varphi_0(z)+\frac{1}{2}\,l\,z\overline{\varphi_0'(z)}+\overline{\psi_0(z)},$$

where

$$\varphi_0(z)=i\varepsilon Rz+(A-2E)^{-1}\gamma, \psi_0(z)=-(2\mu)^{-1}\overline{\gamma}. \quad (4.17)$$

Here R is an arbitrary real constant, $\gamma=(\gamma_1,\gamma_2)^T$ is an arbitrary constant complex vector, and $\varepsilon=(\varepsilon_1,\varepsilon_2)^T$ is the real vector defined by, (see[5]),

$$\varepsilon_1=\frac{1}{\triangle_2}[A_2-H_0(2-A_4)], \quad \varepsilon_2=\frac{1}{\triangle_2}(2-A_1-H_0A_3). \quad (4.18)$$

$$H_0=\frac{A_2(\mu_2+\mu_3)-(2-A_1)(\mu_1+\mu_3)}{(2-A_4)(\mu_2+\mu_3)-A_3(\mu_1+\mu_3)}; \triangle_2=det(A-2E)>0.$$

Due to (4.17) and (4.14) we arrive at

$$\varphi_0(z)=(A-2E)^{-1}\gamma,\ \psi_0(z)=-(2\mu)^{-1}\overline{\gamma},\ z\in G^+. \quad (4.19)$$

Finally comparing (4.11), (4.12) and (4.19) we obtain

$$\gamma=\frac{1}{2\pi i}\int_L\frac{g_0(t)dt}{t-z}+(A-2E)\sum_{j=1}^{p}\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0}{z-z_j}, \quad (4.20)$$

$$-\overline{\gamma}=\frac{1}{2\pi i}\int_L\frac{\overline{g_0(t)}dt}{t-z}-\frac{H}{2\pi i}\int_L\frac{\bar t g_0'(t)dt}{t-z}+\sum_{j=1}^{p}B\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0}{z-z_j}. \quad (4.21)$$

Introduce the notation

$$i\varphi^*(t)=(A-2E)^{-1}g_0(t)+\sum_{j=1}^{p}\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0}{t-z_j}-(A-2E)^{-1}\gamma, \quad (4.22)$$

$$i\psi^*(t)=(2\mu)^{-1}\overline{g_0(t)}-(2\mu)^{-1}H\bar t g_0'(t)$$
$$+(2\mu)^{-1}\sum_{j=1}^{p}B\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0}{t-z_j}+(2\mu)^{-1}\overline{\gamma}\,. \quad (4.23)$$

By (4.20) and (4.21) we obtain

$$\frac{1}{2\pi i}\int_L \frac{\varphi^*(t)dt}{t-z}=0,\ \frac{1}{2\pi i}\int_L \frac{\psi^*(t)dt}{t-z}=0, \forall z\in G^+. \tag{4.24}$$

From (4.24) we have, (see [6, §74]) the vector-functions (4.22) and (4.23) are the boundary value of the vector-functions $\varphi^*(z)$ and $\psi^*(z)$ which are holomorphic in the domains $G_1, G_2, ..., G_{p+1}$ and $\varphi^*(\infty)=\psi^*(\infty)=0$.

After eliminating $g_0(t)$ in (4.22) and (4.23) we obtain

$$(A-2E)\overline{\varphi^*(t)}+B\bar{t}\varphi^{*\prime}(t)+2\mu\psi^*(t)=i\sum_{j=1}^{p}[(A-2E)\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0}{\bar{t}-\bar{z}}$$

$$-B\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0}{t-z_j}+B\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0\bar{t}}{(t-z_j)^2}]-2i\overline{\gamma}, \quad on \quad L. \tag{4.25}$$

Multiplying (4.25) by $\begin{pmatrix}1\\1\end{pmatrix}dt$ and integrating over $L_k, k=\overline{1,p}$. Owing to (2.5) we obtain

$$\begin{pmatrix}B_1+B_3\\B_2+B_4\end{pmatrix}\int_{L_k}[\overline{\varphi^*(t)}dt-\varphi^*(t)\overline{dt}]$$

$$=i\sum_{j=1}^{p}\begin{pmatrix}B_1+B_3\\B_2+B_4\end{pmatrix}M_j^0\begin{pmatrix}1\\1\end{pmatrix}\int_{L_k}\left[\frac{dt}{\bar{t}-\overline{z_i}}+\frac{\overline{dt}}{t-z_i}\right]-4\pi M_k^0, k=\overline{1,p}.$$

Since $M_k^0, (k=\overline{1,p})$ are real constants (see (4.13)) therefore from the last relation it follows

$$M_k^0=0, (k=\overline{1,p}) \tag{4.26}$$

Thus, we have

$$(A-2E)\overline{\varphi^*(t)}+B\bar{t}\varphi^{*\prime}(t)+2\mu\psi^*(t)=-2i\overline{\gamma}, on L_k, k=\overline{1,p+1.}$$

By $(2.4)'$ this condition corresponds to the second boundary value problem of statics in the domains $G_1, G_2, G_3, ..., G_p$, and G_{p+1}, when the boundaries are free form external forces.

By virtue of uniqueness theorem [1] for domain G_{p+1} and the fact that $\varphi^*(\infty)=\psi^*(\infty)=0$, we find that $\varphi^*(z)=\psi^*(z)=0, in\ G_{p+1}$, then $\gamma=0$.

Due to the above reasoning we can write

$$(A-2E)\overline{\varphi^*(t)}+B\bar{t}\varphi^{*\prime}(t)+2\mu\psi^*(t)=0, on\ L_k,\ k=\overline{1,p}.$$

Using the uniqueness theorem for the problem $(II)_0^+$, (see [1]), in the domain $G_k,\ k=\overline{1,p}$ we find that

$$\varphi^*(z)=iR_k\varepsilon z+(A-2E)^{-1}C_k,$$

$$\psi^*(z)=-(2\mu)^{-1}\overline{C_k} \quad z\in G_k, \quad k=\overline{1,p}, \tag{4.27}$$

where R_k is an arbitrary real constant, $C_k = (C_{k1}, C_{k2})^T$ is an arbitrary complex constant vector and $\varepsilon = (\varepsilon_1, \varepsilon_2)^T$ is a real vector defined by (4.18).

From (4.27) it follows, (see (4.22), (4.23) and (4.26)) that

$$g_0(t) = -R_k \varepsilon t + i(A - 2E)^{-1} C_k \quad on \quad L_k,\ k = \overline{1,p},$$

further since $\varphi^*(z) = \psi^*(z) = 0$ in G_{p+1}, therefore

$$g_0(t) = 0 \quad on \quad L_{p+1}.$$

Finally, note that from (4.9), (4.26), (4.7) and (4.16) it follows that $R_k = C_k = 0$ for every k, hence $g_0(t) = 0$ on L.

Thus, we proved that the homogeneous equation correspond to equation (4.8) has no solution different from zero.

Therefore equation (4.8) has one and only one solution $g = (g_1, g_2)^T$. Further note that $g \in C^{o,\alpha}(L)$.

On substituting value $g = (g_1, g_2)^T$ info formula (4.3) and (4.4) we find the analytic vector-functions $\varphi(z)$ and $\psi(z)$.

Having found the vector-functions $\varphi(z)$ and $\psi(z)$ by virtue of (2.3) we obtain a solution of the second boundary value problem provided that the requirement for the principal vector and the principal moment of external forces to be equal to zero is fulfilled. Displacement U is defined to within rigind displacement, while stresses are defined precisely.

The existence of solution of the second boundary value problem can also be proved when domain G is an infinite multiply-connected domain.

REFERENCES

1. Basheleishvili M. Two-dimensional boundary-value problems of statics of the theory of elastic mixtures. *Mem. Differential Equations Math. Phys.*, **6** (1995), 59-105.

2. Basheleishvili M. Analogues of the Kolosov-Muskhelishviuli general representation formulas and Cauchy-Riemann conditions in the theory of elastic mixtures. *Georgian Math. J.*, **4**, 3 (1997), 223-242.

3. Basheleishvili M. Application of analogues of general Kolosov-Muskhelishvili representations in theory of elastic mixtures. *Georgian Math. J.*, **6**, 1 (1999), 1-18.

4. Basheleishvili M. Svanadze K. A new method of solving the basic plane boundary value problems of statics of the elastic mixture theory. *Georgian Math. J.*, **8**, 3 (2001), 427-446.

5. Basheleishvili M. Zazashvili Sh. The basic mixed plane boundary value problem of statics of the elastic mixture theory. *Georgian Math. J.* **7**, 3 (2000), 427-440.

6. Muskhelishvili N.J. Some basic problems of the mathematical theory of elasticity. (Russian) *Nauka, Moscow*, 1966.

7. Natroshvili D., Jagmaidze A.Ya., Svanadze M. Some problems of the linear theory of elastic mixtures. (Russian) *Tbilisi Univ. Press, Tbilissi*, 1986.

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