NEUMANN TYPE INTERIOR BOUNDARY VALUE PROBLEM OF THERMOELASTOSTATICS FOR HEMITROPIC SOLIDS

Ivanidze D., Natroshvili D.

Abstract. The purpose of this paper is investigation of the three-dimensional interior Neumann type boundary value problem of the theory of thermoelastostatics for hemitropic solids. Hemitropic solids belong to the class of Cosserat type continua and the corresponding system of partial differential equations generates a 7×7 nonselfadjoint matrix elliptic operator. The uniqueness and existence results are studied by the potential method and the theory of singular integral equations. The boundary integral operators associated with the layer potentials are analyzed and on the basis of the results obtained we derive the explicit necessary and sufficient conditions for the interior Neumann type boundary value problem to be solvable. We show that solutions are representable in the form of the single layer potential.

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1. Introduction

In a generalized solid continuum, the usual displacement field has to be supplemented by a microrotation field. Such materials are called micropolar or Cosserat solids. They model continua with a complex inner structure whose material particles have 6 degree of freedom (3 displacement components and 3 microrotation components). Recall that the classical elasticity theory allows only 3 degrees of freedom (3 displacement components).

Mathematical models describing the so called hemitropic properties of elastic materials have been proposed by Aero and Kuvshinski [1], [2] (for historical notes see also [3], [4], [19], and the references therein).

Hemitropic solids are not isotropic with respect to inversion, i.e., they are isotropic with respect to all proper orthogonal transformations but not with respect to mirror reflections.

In the present paper we deal with the model of micropolar elasticity for hemitropic solids when the thermal effects are taken into consideration.

In the mathematical theory of hemitropic thermoelasticity there are introduced the asymmetric force stress tensor and couple stress tensor, which are kinematically related with the asymmetric strain tensor, torsion (curvature) tensor and the temperature function via the constitutive equations. All these quantities along with the heat flux vector are expressed in terms of the components of the displacement and microrotation vectors and the temperature function. In turn, the displacement and microrotation vectors and the temperature distribution function satisfy a coupled complex system of second order partial differential equations. When the mechanical and thermal characteristics (displacements, microrotations, temperature, body force, body couple vectors, and heat source) do not depend on the time variable t we have the differential equations of statics. These equations generate a strongly elliptic, formally nonselfadjoint 7×7 matrix differential operator.

The Dirichlet, Neumann and mixed type boundary value problems (BVP) for the so called *pseudo oscillation case* with complex frequency parameter, which are related to the dynamical equations via the Laplace transform, are well investigated for homogeneous bodies of arbitrary shape (see [14], [15], [17], [18], [13], [16] and the references therein).

The main goal of the present paper is investigation of the interior Neumann type boundary value problem of statics of thermoelasticity for hemitropic solids. In the case of static problems there arise significant difficulties which need a special consideration.

Here we develop the boundary integral equations method to obtain the existence and uniqueness results in Hölder $(C^{k,\alpha})$ functional spaces. We reduce the Neumann type BVP to the equivalent system of normally solvable singular integral equations. We construct explicitly the null spaces of the corresponding singular integral operator and its adjoint one, and on the basis of the results obtained we derive necessary and sufficient conditions for the interior Neumann type BVP to be solvable.

2. Problems setting, Green's formulas and uniqueness theorems

Let $\Omega^+ \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial \Omega^+ =: S \in C^{1,\kappa}$ with $0 < \kappa \leq 1$, $\overline{\Omega^+} = \Omega^+ \cup S$, and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. The outward unit normal vector to S at the point $x \in S$ we denote by $n(x) = (n_1(x), n_2(x), n_3(x))$. We assume that the domains Ω^+ are filled with a hemitropic elastic continua.

The basic governing homogeneous equations of the theory of thermoelastostatics for hemitropic materials read as (see [19])

$$(\mu + \alpha)\Delta u(x) + (\lambda + \mu - \alpha) \operatorname{grad} \operatorname{div} u(x) + (\chi + \nu)\Delta\omega(x) + (\delta + \chi - \nu) \operatorname{grad} \operatorname{div} \omega(x) + 2\alpha \operatorname{curl} \omega(x) - \eta \operatorname{grad} \vartheta(x) = 0, (\chi + \nu)\Delta u(x) + (\delta + \chi - \nu) \operatorname{grad} \operatorname{div} u(x) + 2\alpha \operatorname{curl} u(x) + (\gamma + \varepsilon)\Delta\omega(x) + (\beta + \gamma - \varepsilon) \operatorname{grad} \operatorname{div} \omega(x) + 4\nu \operatorname{curl} \omega(x) - \zeta \operatorname{grad} \vartheta(x) - 4\alpha\omega(x) = 0, \kappa' \Delta \vartheta(x) = 0,$$
(2.1)

where $u = (u_1, u_2, u_3)^{\top}$ and $\omega = (\omega_1, \omega_2, \omega_3)^{\top}$ are the displacement vector and the microrotation vector respectively, ϑ is the temperature distribution function, α , β , γ , δ , λ , μ , ν , χ , ε , η , ζ and κ' are the material constants, $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, j = 1, 2, 3, the symbol $(\cdot)^{\top}$ denotes transposition.

The matrix differential operator generated by these equations is not formally selfadjoint and has the form

$$L(\partial) = \begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial) & L^{(5)}(\partial) \\ L^{(3)}(\partial) & L^{(4)}(\partial) & L^{(6)}(\partial) \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa' \Delta \end{bmatrix}_{7\times 7},$$
 (2.2)

where

$$L^{(1)}(\partial) := (\mu + \alpha)\Delta I_3 + (\lambda + \mu - \alpha)Q(\partial),$$

$$L^{(2)}(\partial) = L^{(3)}(\partial) := (\chi + \nu)\Delta I_3 + (\delta + \chi - \nu)Q(\partial) + 2\alpha R(\partial),$$

$$L^{(4)}(\partial) := [(\gamma + \varepsilon)\Delta - 4\alpha]I_3 + (\beta + \alpha - \varepsilon)Q(\partial) + 4\nu R(\partial),$$

$$L^{(5)}(\partial) := -\eta \nabla^{\top}, \quad L^{(6)}(\partial) := -\zeta \nabla^{\top},$$

$$R(\partial) := [-\varepsilon_{pqj}\partial_j]_{3\times 3}, \quad Q(\partial) := [\partial_k\partial_j]_{3\times 3}.$$

(2.3)

Here and in what follows ε_{pqj} denotes the permutation (Levi-Civitá) symbol and I_k stands for the $k \times k$ unit matrix. Throughout the paper repeated indices indicate summation from one to three if not otherwise stated.

Denote by $L^*(\partial) := L^{\top}(-\partial)$ the operator formally adjoint to $L(\partial)$. Moreover, let $\widetilde{L}(\partial)$ denote the operator corresponding to the equilibrium equations of hemitropic elastostatics when thermal effects are not taken into consideration (see [14])

$$\widetilde{L}(\partial) = \begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial) \\ L^{(3)}(\partial) & L^{(4)}(\partial) \end{bmatrix}_{6\times 6}^{6}, \qquad (2.4)$$

where $L^{(k)}(\partial)$ are defined in (2.3). Note that $\widetilde{L}(\partial)$ is formally selfadjoint, i.e., $\widetilde{L}(\partial) = \widetilde{L}^*(\partial) = \widetilde{L}^\top(-\partial)$.

The force stress tensor $\{\tau_{pq}\}_{3\times 3}$ and the couple stress tensor $\{\mu_{pq}\}_{3\times 3}$ in the linear theory of hemitropic thermoelasticity read as follows (the constitutive equations) [18]

$$\begin{aligned} \tau_{pq} &= \tau_{pq}(U) := (\mu + \alpha)\partial_p u_q + (\mu - \alpha)\partial_q u_p + \lambda \delta_{pq} \operatorname{div} u + \delta \,\delta_{pq} \operatorname{div} \omega \\ &+ (\varkappa + \nu)\partial_p \omega_q + (\varkappa - \nu)\partial_q \omega_p - 2\alpha \varepsilon_{pqk}\omega_k - \delta_{pq} \,\eta \,\vartheta, \\ \mu_{pq} &= \mu_{pq}(U) := \delta \,\delta_{pq} \operatorname{div} u + (\varkappa + \nu) \Big[\partial_p u_q - \varepsilon_{pqk}\omega_k\Big] + \beta \,\delta_{pq} \operatorname{div} \omega \\ &+ (\varkappa - \nu) \Big[\partial_q u_p - \varepsilon_{qpk}\omega_k\Big] + (\gamma + \varepsilon)\partial_p \omega_q + (\gamma - \varepsilon)\partial_q \omega_p - \delta_{pq} \zeta \vartheta, \end{aligned}$$

where $U = (u, \omega, \vartheta)^{\top}$, δ_{pq} is the Kronecker delta.

The components of the force stress vector $\tau^{(n)}$ and the couple stress vector $\mu^{(n)}$, acting on a surface element with a unite normal vector $n = (n_1, n_2, n_3)$, are expressed as

$$\tau^{(n)} = \left(\tau_1^{(n)}, \, \tau_2^{(n)}, \, \tau_3^{(n)}\right)^\top, \qquad \mu^{(n)} = \left(\mu_1^{(n)}, \, \mu_2^{(n)}, \, \mu_3^{(n)}\right)^\top,$$

where

$$\tau_q^{(n)} = \tau_{pq} n_p, \qquad \mu_q^{(n)} = \mu_{pq} n_p, \quad q = 1, 2, 3.$$

Introduce the generalized stress operators associated with the differential operators $L(\partial)$ and $\widetilde{L}(\partial)$ (cf. [14], [17], [18])

$$\mathcal{P}(\partial, n) = \begin{bmatrix} T^{(1)}(\partial, n) & T^{(2)}(\partial, n) & -\eta n^{\top} \\ T^{(3)}(\partial, n) & T^{(4)}(\partial, n) & -\zeta n^{\top} \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa' \partial_n \end{bmatrix}_{7\times 7},$$
(2.5)
$$\mathcal{P}^*(\partial, n) = \begin{bmatrix} T^{(1)}(\partial, n) & T^{(2)}(\partial, n) & [0]_{3\times 1} \\ T^{(3)}(\partial, n) & T^{(4)}(\partial, n) & [0]_{3\times 1} \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa' \partial_n \end{bmatrix}_{7\times 7},$$
(2.6)

where

$$T^{(j)} = [T^{(j)}_{pq}]_{3\times3}, \quad j = \overline{1,4}, \quad n = (n_1, n_2, n_3),$$

$$T^{(1)}_{pq}(\partial, n) = (\mu + \alpha)\delta_{pq}\partial_n + (\mu - \alpha)n_q\partial_p + \lambda n_p\partial_q,$$

$$T^{(2)}_{pq}(\partial, n) = (\chi + \nu)\delta_{pq}\partial_n + (\chi - \nu)n_q\partial_p + \delta n_p\partial_q - 2\alpha\varepsilon_{pqk}n_k,$$

$$T^{(3)}_{pq}(\partial, n) = (\chi + \nu)\delta_{pq}\partial_n + (\chi - \nu)n_q\partial_p + \delta n_p\partial_q,$$

$$T^{(4)}_{pq}(\partial, n) = (\gamma + \varepsilon)\delta_{pq}\partial_n + (\gamma - \varepsilon)n_q\partial_p + \beta n_p\partial_q - 2\nu\varepsilon_{pqk}n_k.$$

Here $\partial_n = \partial/\partial n$ denotes the usual normal derivative.

In addition, let us introduce the "pure hemitropic boundary stress operator" associated with the differential operator $\widetilde{L}(\partial)$

$$T(\partial, n) = \begin{bmatrix} T^{(1)}(\partial, n) & T^{(2)}(\partial, n) \\ T^{(3)}(\partial, n) & T^{(4)}(\partial, n) \end{bmatrix}_{6 \times 6}$$
(2.7)

with $T^{(j)}(\partial, n)$ defined in (2.5).

For a vector $U = (u, \omega, \vartheta)^{\top}$ the seven vector $\mathcal{P}(\partial, n) U$ has the following physical sense: the first three components

$$T^{(1)}(\partial, n)u + T^{(2)}(\partial, n)\omega - \eta n^{\top} \vartheta = \left(\tau_1^{(n)}, \tau_2^{(n)}, \tau_3^{(n)}\right)^{\top}$$

correspond to the thermo-mechanical stress vector, the second triplet

$$T^{(3)}(\partial, n)u + T^{(4)}(\partial, n)\omega - \zeta n^{\top} \vartheta = \left(\mu_1^{(n)}, \, \mu_2^{(n)}, \, \mu_3^{(n)}\right)^{\top}$$

corresponds to the thermo-mechanical couple stress vector, while the seventh component $\kappa' \partial_n \vartheta$ corresponds to the normal component of the heat flux vector.

For regular vector-functions

$$U = (u, \omega, \vartheta)^{\top}, \, U' = (u', \omega', \vartheta')^{\top} \in [C^2(\Omega^+)]^7 \cap [C^1(\overline{\Omega^+})]^7,$$

the following Green's formula holds [18]

$$\int_{\Omega^+} \left[U' \cdot L(\partial)U - L^*(\partial)U' \cdot U \right] dx = \int_{\partial\Omega^+} \left[\{U'\}^+ \cdot \{\mathcal{P}(\partial, n)U\}^+ - \{\mathcal{P}^*(\partial, n)U'\}^+ \cdot \{U\}^+ \right] dS,$$
(2.8)

where the operator $L(\partial)$ is defined in (2.2) and $L^*(\partial) = L^{\top}(-\partial)$ is the operator formally adjoint to $L(\partial)$, while $\mathcal{P}(\partial, n)$ and $\mathcal{P}^*(\partial, n)$ are given by (2.5) and (2.6); the symbols $\{\cdot\}^{\pm}$ denote one sided limits on S from Ω^{\pm} respectively, while the central dot denotes scalar product of two vectors in Euclidean space \mathbb{R}^n .

3. Problem formulation and uniqueness theorem

The Neumann type interior boundary value problem $(N)^+$ is formulated as follows: Find a regular vector-function $U \in [C^1(\overline{\Omega^+})]^7 \cap [C^2(\Omega^+)]^7$ satisfying the differential equation

$$L(\partial)U(x) = 0, \qquad x \in \Omega^+ \tag{3.1}$$

and the Neumann type boundary condition on S

$$\left\{\mathcal{P}(\partial, n)U(x)\right\}^{+} = F(x), \quad x \in S,$$
(3.2)

where $F = (F_1, F_2, ..., F_7)^{\top} \in [C(S)]^7$ is a given vector-function. The following uniqueness theorem holds true.

Theorem 3.1. A general solution to the homogeneous Neumann type interior boundary value problem reads as

$$U_0 = (\widetilde{\Psi}, 0)^\top + \vartheta_0 \left(u_0, \omega_0, 1 \right)$$

where $\widetilde{\Psi}$ is a generalized rigid displacement vector,

$$\widetilde{\Psi}(x) = ([a \times x] + b, a)^{\top}$$
(3.3)

with $a = (a_1, a_2, a_3)^{\top}$ and $b = (b_1, b_2, b_3)^{\top}$ being arbitrary three dimensional constant vectors, ϑ_0 is an arbitrary constant, while the vector-functions $u_0 = (u_{01}, u_{02}, u_{03})^{\top}$ and $\omega_0 = (\omega_{01}, \omega_{02}, \omega_{03})^{\top}$ are such that the six dimensional vector-function $\widetilde{V}_0 = (u_0, \omega_0)^{\top}$ solves the following boundary value problem

$$\widetilde{L}(\partial)\widetilde{V}_0(x) = 0, \qquad x \in \Omega^+,$$

$$\{T(\partial, n)\widetilde{V}_0\}^+ = (\eta \, n(x), \zeta \, n(x))^\top, \qquad x \in S.$$
(3.4)

Here η and ζ are material parameters involved in the basic system (2.1) and the operators $\widetilde{L}(\partial)$ and $T(\partial, n)$ are defined in (2.4) and (2.7).

Proof. Form the structure of the operators (2.2) and (2.5) it is easy to see that for the temperature function ϑ the corresponding boundary value problem can be separated, which reads as

$$\Delta \vartheta(x) = 0, \qquad x \in \Omega^+,$$

$$\left\{\frac{\partial\vartheta(x)}{\partial n}\right\}^+ = 0, \qquad x \in S.$$

A general solution to this problem is a constant function,

$$\vartheta(x) = \vartheta_0 = const, \qquad x \in \Omega^+,$$

where ϑ_0 is an arbitrary real constant.

Therefore a general solution to the homogeneous Neumann type boundary value problem has the following form: $U = (u, \omega, \vartheta_0)^\top = (\widetilde{U}, \vartheta_0)^\top$ with $\widetilde{U} = (u, \omega)^\top$. Consequently, in view of (2.2), (2.4), (2.5), and (2.7), the vector \widetilde{U} solves the following nonhomogeneous boundary value problem

$$\widetilde{L}(\partial) \widetilde{U}(x) = 0, \qquad x \in \Omega^+,$$

$$\{T(\partial, n) \widetilde{U}(x)\}^+ = \widetilde{F}_0(x), \qquad x \in S,$$
(3.5)

where

$$\widetilde{F}_0(x) = \vartheta_0 \left(\eta \, n(x), \zeta \, n(x) \right)^\top, \qquad x \in S.$$
(3.6)

Recall that n(x) is the outward unit normal vector at the point $x \in S$, while η and ζ are the material parameters. Thus \widetilde{U} is a solution to the nonhomogeneous interior Neumann type boundary value problem for hemitropic model, when the thermal effects are not taken into consideration. In the reference [18] it is shown that the condition

$$\int_{S} \widetilde{F}_{0}(x) \cdot \widetilde{\Psi}(x) \, dS = 0 \tag{3.7}$$

is necessary and sufficient for the problem (3.5)-(3.6) to be solvable. Here Ψ is a generalized rigid displacement vector define in (3.3).

With the help of the relations

$$[a \times x] \cdot n = [x \times n] \cdot a, \quad \int_{S} n_k(x) dS = 0, \quad \int_{S} [x_j n_k(x) - x_k n_j(x)] dS = 0, \quad k, j = 1, 2, 3,$$

and the Gauss divergence theorem, it is easy to verify that conditions (3.7) for the vector (3.6) hold true,

$$\int_{S} \vartheta_{0}(\eta n(x), \zeta n(x))^{\top} \cdot ([a \times x] + b, a)^{\top} dS = \vartheta_{0} \int_{S} \left\{ \eta \left(n \cdot [a \times x] + n \cdot b \right) + \zeta n \cdot a \right\} dS$$
$$= \vartheta_{0} \eta \int_{S} [x \times n] \cdot a \, dS = \vartheta_{0} \eta \sum_{k=1}^{3} a_{k} \int_{S} [x \times n]_{k} dS = 0.$$

Consequently, the boundary value problem (3.5) is solvable for arbitrary constant ϑ_0 and solutions are defined modulo the vector $\widetilde{\Psi}$ given by (3.3). Denote by $\widetilde{V}_0 := (u_0, \omega_0)^{\top}$ with $u_0 = (u_{01}, u_{02}, u_{03})^{\top}$ and $\omega_0 = (\omega_{01}, \omega_{02}, \omega_{03})^{\top}$ some particular solution of problem (3.4) which coincide with problem (3.5) for $\vartheta_0 = 1$. Then it follows that $\vartheta_0 \widetilde{V}_0$ represents a particular solution of problem (3.5), while a general solution of the same problem reads as $\widetilde{U} = \vartheta_0 \widetilde{V}_0 + \widetilde{\Psi}$. Whence we deduce that the vector $U = (\widetilde{U}, \vartheta_0)^{\top} = \vartheta_0 (u_0, \omega_0, 1)^{\top} + (\widetilde{\Psi}, 0)$ is a general solution to the homogeneous interior Neumann type problem which completes the proof.

Remark 3.2. Introduce the system of vector-functions $\{\Phi^{(k)}(x)\}_{k=1}^7$, where

$$\Phi^{(1)} = (0, -x_3, x_2, 1, 0, 0, 0)^{\top}, \quad \Phi^{(2)} = (x_3, 0, -x_1, 0, 1, 0, 0)^{\top},$$

$$\Phi^{(3)} = (-x_2, x_1, 0, 0, 0, 1, 0)^{\top}, \quad \Phi^{(4)} = (1, 0, 0, 0, 0, 0, 0)^{\top},$$

$$\Phi^{(5)} = (0, 1, 0, 0, 0, 0, 0)^{\top}, \qquad \Phi^{(6)} = (0, 0, 1, 0, 0, 0, 0)^{\top},$$

$$\Phi^{(7)} = (u_0, \omega_0, 1)^{\top}.$$
(3.8)

Here the vector $(u_0, \omega_0)^{\top}$ is a particular solution of the nonhomogeneous problem (3.4) existence of which is shown in the above presented proof of Theorem 3.1 It is easy to check that the vectors (3.8) are linearly independent in Ω^+ and each of them is a solution to the homogeneous interior Neumann type problem (3.1)–(3.2) with F = 0. Moreover, from Theorem 3.1 it follows that a general solution to the homogeneous interior Neumann type problem (3.1)–(3.2) with F = 0.

$$U(x) = \sum_{k=1}^{7} C_k \Phi^{(k)}(x),$$

where C_k are arbitrary real constants, while $\Phi^{(k)}(x)$ are defined in (3.8).

In our analysis below, we need uniqueness results for the exterior boundary value problems for the operators $L(\partial)$ and $L^*(\partial)$ in special spaces of vector-functions which are bounded at infinity. To this end let us introduce the following definitions.

Definition 3.3. A vector-function $U = (u, \omega, \vartheta)^{\top}$ is said to belong to the class $Z(\Omega^{-})$ if it is continuous in a neighbourhood of infinity and satisfies the following asymptotic conditions

(i)
$$u(x) = \mathcal{O}(1), \quad \omega(x) = \mathcal{O}(|x|^{-2}), \quad \vartheta(x) = \mathcal{O}(|x|^{-1}) \quad \text{as} \quad |x| \to \infty,$$

(ii) $\lim_{R \to \infty} \frac{1}{4\pi R^2} \int_{\Sigma(0,R)} u(x) \, d\Sigma(0,R) = 0,$

where $\Sigma(0, R)$ is a sphere centered at the origin and radius R.

Definition 3.4. A vector-function $U^* = (u^*, \omega^*, \vartheta^*)^{\top}$ is said to belong to the class $Z^*(\Omega^-)$ if it is continuous in a neighbourhood of infinity and satisfies the following asymptotic conditions

(i)
$$u^*(x) = \mathcal{O}(|x|^{-1}), \quad \omega^*(x) = \mathcal{O}(|x|^{-2}), \quad \vartheta^*(x) = \mathcal{O}(1) \quad \text{as} \quad |x| \to \infty(3.9)$$

(ii)
$$\lim_{R \to \infty} \frac{1}{4\pi R^2} \int_{\Sigma(0,R)} \vartheta^*(x) \, d\Sigma(0,R) = 0.$$
(3.10)

4. Layer potentials and general integral representations

The matrix of fundamental solutions $\Gamma(x-y) = [\Gamma_{kj}(x-y)]_{7\times7}$ associated with the operator $L(\partial)$ can be constructed explicitly in terms of standard functions (see Appendix). It is a solution of the distributional equation $L(\partial_x)\Gamma(x-y) = I_7\delta(x-y)$, where $\delta(x-y)$ is Dirac's delta distribution. Let us introduce the single layer and double layer potentials

$$V(g)(x) = V_S(g)(x) := \int_S \Gamma(x - y) g(y) \, dS_y, \quad x \in \mathbb{R}^3 \setminus S,$$
$$W(g)(x) = W_S(g)(x) := \int_S \left[\mathcal{P}^* \left(\partial_y, n(y) \right) \Gamma^\top (x - y) \right]^\top g(y) \, dS_y, \quad x \in \mathbb{R}^3 \setminus S,$$

where $g = (g_1, g_2, ..., g_7)^{\top}$ and $h = (h_1, h_2, ..., h_7)^{\top}$ are density vector-functions defined on S, while the boundary operator $\mathcal{P}^*(\partial, n)$ is defined in (2.6).

Further, we introduce the "adjoint" layer potentials associated with the operator $L^*(\partial)$,

$$V^*(g)(x) := \int_{S} \Gamma^*(x-y) g(y) \, dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{4.1}$$

$$W^*(g)(x) := \int_{S} \left[\mathcal{P}(\partial_y, n(y)) \left[\Gamma^*(x-y) \right]^\top \right]^\top g(y) \, dS_y, \quad x \in \mathbb{R}^3 \setminus S, \qquad (4.2)$$

where $\Gamma^*(x-y) := \Gamma^\top(y-x)$ is a fundamental matrix of the operator $L^*(\partial)$, the boundary operator $\mathcal{P}(\partial, n)$ is defined in (2.5), and $g = (g_1, g_2, ..., g_7)^\top$ and $h = (h_1, h_2, ..., h_7)^\top$ are density vector-functions defined on S.

Theorem 4.1. Let $S \in C^{1,\kappa}$ with $0 < \kappa \leq 1$ and vector-functions $U \in [C^1(\overline{\Omega^+})]^7 \cap [C^2(\Omega^+)]^7$ and $U^* \in [C^1(\overline{\Omega^+})]^7 \cap [C^2(\Omega^+)]^7$ be regular solutions of the equations $L(\partial)U = 0$ and $L^*(\partial)U^* = 0$ in Ω^+ respectively. Then the following integral representation formulas hold

$$W(\{U\}^{+})(x) - V(\{\mathcal{P}U\}^{+})(x) = \begin{cases} U(x), & x \in \Omega^{+}, \\ 0, & x \in \Omega^{-}, \end{cases}$$
(4.3)

$$W^*(\{U^*\}^+)(x) - V^*(\{\mathcal{P}^*U^*\}^+)(x) = \begin{cases} U^*(x), & x \in \Omega^+, \\ 0, & x \in \Omega^-. \end{cases}$$
(4.4)

Proof. It is standard and follows from Green's formula (2.8).

The mapping properties of the above introduced layer potentials V, W, V^* , and W^* can be established by standard arguments applied, e.g., in the references [9], [10], [6], [12], [14].

Theorem 4.2. The single and double layer potentials V(g) and W(g) solve the homogeneous equation $L(\partial)U = 0$ in $\mathbb{R}^3 \setminus S$, belong to the class $Z(\Omega^-)$ and the following

operators

$$V : [C^{k,\sigma}(S)]^7 \to [C^{k+1,\sigma}(\overline{\Omega^{\pm}})]^7,$$

$$W : [C^{k,\sigma}(S)]^7 \to [C^{k,\sigma}(\overline{\Omega^{\pm}})]^7,$$

are continuous provided $S \in C^{k+1,\kappa}$, where $k \ge 0$ is an integer and $0 < \sigma < \kappa \le 1$. **Proof.** It can be found in [7].

Lemma 4.3. The single and double layer potentials $V^*(g)$ and $W^*(g)$ solve the homogeneous equation $L^*(\partial)U^* = 0$ in $\mathbb{R}^3 \setminus S$, belong to the class $Z^*(\Omega^-)$, and the following operators

$$V^* : [C^{k,\sigma}(S)]^7 \to [C^{k+1,\sigma}(\overline{\Omega^{\pm}})]^7,$$
$$W^* : [C^{k,\sigma}(S)]^7 \to [C^{k,\sigma}(\overline{\Omega^{\pm}})]^7$$

are continuous provided $S \in C^{k+1,\kappa}$, where $k \ge 0$ is an integer number and $0 < \sigma < \kappa \le 1$.

Proof. It can be found in [8]. **Theorem 4.4.** Let $S \in C^{1,\kappa}$, $g \in [C^{0,\sigma}(S)]^7$ and $h \in [C^{1,\sigma}(S)]^7$ with $0 < \sigma < \kappa \leq 1$. Then the following relations hold true:

$$\begin{aligned} \{V(g)(x)\}^{\pm} &= V(g)(x) = \mathcal{H}g(x), \\ \{\mathcal{P}(\partial_x, n(x))V(g)(x)\}^{\pm} &= [\mp 2^{-1}I_7 + \mathcal{K}] g(x), \\ \{W(g)(x)\}^{\pm} &= [\pm 2^{-1}I_7 + \mathcal{N}] g(x), \\ \{\mathcal{P}(\partial_x, n(x))W(h)(x)\}^+ &= \{\mathcal{P}(\partial_x, n(x))W(h)(x)\}^- = \mathcal{L}h(x), \quad S \in C^{2,\kappa}, \end{aligned}$$

where \mathcal{H} is a weakly singular integral operator, \mathcal{K} and \mathcal{N} are singular integral operators, while \mathcal{L} is a singular integro-differential operator

$$\begin{aligned} \mathcal{H}g(x) &:= \int_{S} \Gamma(x-y)g(y)dS_{y}, \\ \mathcal{K}g(x) &:= \int_{S} \left[\mathcal{P}\big(\partial_{x}, n(x)\big)\Gamma(x-y) \right]g(y)dS_{y}, \\ \mathcal{N}g(x) &:= \int_{S} \left[\mathcal{P}^{*}\big(\partial_{y}, n(y)\big)\Gamma^{\top}(x-y) \right]^{\top}g(y)dS_{y}, \\ \mathcal{L}h(x) &:= \lim_{\Omega^{\pm} \ni z \to x \in S} \mathcal{P}\big(\partial_{z}, n(x)\big) \int_{S} \left[\mathcal{P}^{*}\big(\partial_{y}, n(y)\big)\Gamma^{\top}(z-y) \right]^{\top}h(y)dS_{y}. \end{aligned}$$

Proof. It can be found in [7].

Theorem 4.5. Let $k \ge 0$ be integers, and $S \in C^{k+1,\kappa}$ with $0 < \sigma < \kappa \le 1$. Then the following operators are continuous

$$\begin{aligned} \mathcal{H} &: \ [C^{k,\sigma}(S)]^7 \to [C^{k+1,\sigma}(S)]^7, \\ \mathcal{N} &: \ [C^{k,\sigma}(S)]^7 \to [C^{k,\sigma}(S)]^7, \\ \end{aligned}$$

$$\begin{aligned} \mathcal{K} &: \ [C^{k,\sigma}(S)]^7 \to [C^{k,\sigma}(S)]^7, \\ \mathcal{L} &: \ [C^{k,\sigma}(S)]^7 \to [C^{k-1,\sigma}(S)]^7. \end{aligned}$$

Moreover, the operators

$$\pm 2^{-1}I_7 + \mathcal{K} : [C^{k,\sigma}(S)]^7 \to [C^{k,\sigma}(S)]^7, \qquad \pm 2^{-1}I_7 + \mathcal{N} : [C^{k,\sigma}(S)]^7 \to [C^{k,\sigma}(S)]^7$$

are elliptic singular integral operators with index equal to zero. The principal homogenous symbol matrices of the operators $-\mathcal{H}$ and \mathcal{L} are positive definite.

The operators \mathcal{H} , $\pm \frac{1}{2}I_7 + \mathcal{K}$, $\pm \frac{1}{2}I_7 + \mathcal{N}$ and \mathcal{L} are pseudodifferential operators with zero index and of order -1, 0, 0, and 1, respectively.

Moreover, the following operator equalities hold true:

$$\mathcal{NH} = \mathcal{HK}, \quad \mathcal{LN} = \mathcal{KL}, \quad \mathcal{HL} = -4^{-1}I_7 + \mathcal{N}^2, \quad \mathcal{LH} = -4^{-1}I_7 + \mathcal{K}^2.$$

Proof. It can be found in [18].

Remark 4.6. Let $S \in C^{2,\kappa}$ and $0 < \sigma < \kappa \leq 1$. The integral operator

$$\mathcal{H} : [C^{0,\sigma}(S)]^7 \to [C^{1,\sigma}(S)]^7$$

is invertible and

$$[\mathcal{H}]^{-1}$$
 : $[C^{1,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7$

is a pseudodifferential operator of order 1, more precisely, it is a singular integrodifferential operator (cf., [18]).

Now we prove the counterpart of Theorem for exterior unbounded domains.

Theorem 4.7. Let $S \in C^{1,\kappa}$ with $0 < \kappa \leq 1$ and vector-functions $U \in [C^1(\overline{\Omega^+})]^7 \cap [C^2(\Omega^-)]^7 \cap Z(\Omega^-)$ and let $U^* \in [C^1(\overline{\Omega^-})]^7 \cap [C^2(\Omega^+)]^7 \cap Z^*(\Omega^-)$ be regular solutions of the equations $L(\partial)U = 0$ and $L^*(\partial)U^* = 0$ in Ω^- respectively. Then the following integral representation formulas hold

$$-W(\{U\}^{-})(x) + V(\{\mathcal{P}U\}^{-})(x) = \begin{cases} U(x), & x \in \Omega^{-}, \\ 0, & x \in \Omega^{+}, \end{cases}$$
(4.5)

$$-W^*(\{U^*\}^-)(x) + V^*(\{\mathcal{P}^*U^*\}^-)(x) = \begin{cases} U^*(x), & x \in \Omega^-, \\ 0, & x \in \Omega^+. \end{cases}$$
(4.6)

Proof. Formula (4.5) is derived in [7]. To prove (4.6) we proceed as follows. Let U^* be as in the theorem and let us write the integral representation formula (4.4) for a bounded domain $\Omega_R^- := \Omega^- \cap B(0, R)$, where R is a sufficiently large positive number, $B(0, R) := \{x \in \mathbb{R}^3 : |x| < R\}$ is a ball centered at the origin and radius R, such that $\overline{\Omega^+} \subset B(0, R)$,

$$U^*(x) = -W^*_S(\{U^*\}^-_S) + V^*_S(\{\mathcal{P}^*U^*\}^-_S) + \Phi^*_R(x), \quad x \in \Omega^-_R,$$
(4.7)

$$0 = -W_S^*(\{U^*\}_S^-) + V_S^*(\{\mathcal{P}^*U^*\}_S^-) + \Phi_R^*(x), \quad x \in \Omega^+ \cup \left[\mathbb{R}^3 \setminus \overline{B(0,R)}\right]; \quad (4.8)$$

here V_S^* and W_S^* are the single and double layer potentials defined in (4.1) and (4.2), while

$$\Phi_R^*(x) := W_{\Sigma_R}^* \big(\{U^*\}_{\Sigma_R}^+ \big)(x) - V_{\Sigma_R}^* \big(\{\mathcal{P}^*U^*\}_{\Sigma_R}^+ \big)(x)$$
(4.9)

with $V_{\Sigma_R}^*$ and $W_{\Sigma_R}^*$ being again the single and double layer potentials with the integration surface $\Sigma_R = \partial B(0, R)$.

From equality (4.9) it follows that

$$L^*(\partial)\Phi^*_R(x) = 0, \quad x \notin \Sigma_R.$$
(4.10)

Moreover, from (4.7) and (4.8) we have

$$\Phi_R^*(x) = U^*(x) + W_S^*(\{U^*\}_S^-) - V_S^*(\{\mathcal{P}^*U^*\}_S^-), \quad x \in \Omega_R^-, \\
\Phi_R^*(x) = W_S^*(\{U^*\}_S^-) - V_S^*(\{\mathcal{P}^*U^*\}_S^-), \quad x \in \Omega^+ \cup \left[\mathbb{R}^3 \setminus \overline{B(0,R)}\right].$$

This implies that for sufficiently large numbers $R_1 < R_2$,

$$\Phi_{R_1}^*(x) = \Phi_{R_2}^*(x) \quad \text{for} \quad |x| < R_1 < R_2.$$
(4.11)

Therefore, for arbitrary $x \in \mathbb{R}^3$ the following limit exists

$$\Phi^*(x) := \lim_{R \to \infty} \Phi^*_R(x) = \begin{cases} U^*(x) + W^*_S(\{U^*\}^-_S)(x) - V^*_S(\{\mathcal{P}^*U^*\}^-_S)(x), & x \in \Omega^-, \\ W^*_S(\{U^*\}^-_S)(x) - V^*_S(\{\mathcal{P}^*U^*\}^-_S)(x), & x \in \Omega^+. \end{cases}$$
(4.12)

Consequently,

$$L^*(\partial)\Phi^*(x) = 0, \qquad x \in \Omega^+ \cup \Omega^-$$

On the other hand, from (4.11) we get

$$\Phi^*(x) = \lim_{R \to \infty} \Phi^*_R(x) = \Phi^*_{R_1}(x)$$
(4.13)

for arbitrary $x \in \mathbb{R}^3$ with $R_1 > |x|$ and $\overline{\Omega^+} \subset B(0, R_1)$. From (4.9) and (4.10) then we conclude

$$L^*(\partial)\Phi^*(x) = 0, \qquad x \in \mathbb{R}^3.$$
(4.14)

At the same time, from (4.12) we have

$$\Phi^* \in Z^*(\mathbb{R}^3),\tag{4.15}$$

since $U^* \in Z^*(\Omega^-)$ and $W^*_S, V^*_S \in Z^*(\Omega^-)$ due to Lemma 4.3.

From the relations (4.14) we deduce that $\Phi^*(x) = 0$, for all $x \in \mathbb{R}^3$. Indeed, from the relations (4.14)-(4.15) by the Fourier transform we get

$$L^*(-i\xi)\,\widehat{\Phi^*}(\xi) = 0, \qquad \xi \in \mathbb{R}^3$$

where $\widehat{\Phi^*}(\xi)$ is a generalized vector-function that belongs to the Schwartz space of tempered distributions. Since the determinant det $L^*(-i\xi)$ is nonsingular for $\xi \in \mathbb{R}^3 \setminus \{0\}$ (see [18]), it follows that the support of the distribution $\widehat{\Phi^*}(\xi)$ is the origin $\xi = 0$. Consequently, $\widehat{\Phi^*}$ is a linear combination of the Dirac distribution and its derivatives,

$$\widehat{\Phi^*}(\xi) = \sum_{|\alpha| \leqslant M} C_{\alpha} \delta^{(\alpha)}(\xi),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, C_{α} are constant seven dimensional vectors, M is a nonnegative integer, while $\delta^{(\alpha)}$ stands for the α -th order derivative of δ . Therefore the vector-function $\Phi^*(x)$ is a polynomial in x,

$$\Phi^*(x) = \sum_{|\alpha| \leqslant M} C_{\alpha} x^{\alpha}, \qquad x \in \mathbb{R}^3.$$

Further, since $\Phi^* \in Z^*(\mathbb{R}^3)$, in accordance with (3.9) and (3.10), we finally conclude $\Phi^*(x) = 0$ for $x \in \mathbb{R}^3$. Now, passing to the limit in (4.7) as $R \to \infty$ and keeping in mind (4.13), we arrive at the general integral representation formula (4.6).

Further we characterize the jump relations for the adjoint layer potentials (for details see [8]).

Theorem 4.8. Let $S \in C^{1,\kappa}$, $g \in [C^{0,\sigma}(S)]^7$ and $h \in [C^{1,\sigma}(S)]^7$ with $0 < \sigma < \kappa \leq 1$. Then for all points $x \in S$ the following relations hold true:

$$\{V^*(g)(x)\}^{\pm} = V^*(g)(x) = \mathcal{H}^*g(x), \tag{4.16}$$

$$\{\mathcal{P}^*(\partial_x, n(x))V^*(g)(x)\}^{\pm} = [\mp 2^{-1}I_7 + \mathcal{K}^*]g(x), \tag{4.17}$$

$$\{W^*(g)(x)\}^{\pm} = [\pm 2^{-1}I_7 + \mathcal{N}^*]g(x), \tag{4.18}$$

$$\{\mathcal{P}^*(\partial_x, n(x))W^*(h)(x)\}^+ = \{\mathcal{P}^*(\partial_x, n(x))W^*(h)(x)\}^- = \mathcal{L}^*h(x), \quad S \in C^{2,\kappa}, \quad (4.19)$$

where the operators \mathcal{H}^* , \mathcal{K}^* , \mathcal{N}^* , and \mathcal{L}^* are pseudodifferential operators of order -1, 0, 0, and 1, respectively, and are defined by the formulas

$$\mathcal{H}^*g(x) := \int_S \Gamma^*(x-y)g(y)dS_y,\tag{4.20}$$

$$\mathcal{K}^*g(x) := \int_S \left[\mathcal{P}^*(\partial_x, n(x)) \Gamma^*(x-y) \right] g(y) dS_y, \tag{4.21}$$

$$\mathcal{N}^*g(x) := \int_S \left[\mathcal{P}(\partial_y, n(y)) \left[\Gamma^*(x-y) \right]^\top \right]^\top g(y) dS_y, \tag{4.22}$$

$$\mathcal{L}^*h(x) := \lim_{\Omega^{\pm} \ni z \to x \in S} \mathcal{P}^*(\partial_z, n(x)) \int_S \left[\mathcal{P}(\partial_y, n(y)) \left[\Gamma^*(z-y) \right]^\top \right]^\top g(y) dS_y.$$
(4.23)

The following equalities hold in appropriate function spaces:

$$\mathcal{N}^* \mathcal{H}^* = \mathcal{H}^* \mathcal{K}^*, \qquad \qquad \mathcal{L}^* \mathcal{N}^* = \mathcal{K}^* \mathcal{L}^*, \\ \mathcal{H}^* \mathcal{L}^* = -4^{-1} I_7 + [\mathcal{N}^*]^2, \qquad \mathcal{L}^* \mathcal{H}^* = -4^{-1} I_7 + [\mathcal{K}^*]^2.$$

Proof. It can be found in [8].

Lemma 4.9. Let $S \in C^{2,\kappa}$ and $0 < \sigma < \kappa \leq 1$. The integral operator

$$\mathcal{H}^* : [C^{0,\sigma}(S)]^7 \to [C^{1,\sigma}(S)]^7$$

is invertible and

$$[\mathcal{H}^*]^{-1}$$
 : $[C^{1,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7$

is a pseudodifferential operator of order 1, more precisely, it is a singular integrodifferential operator.

Proof. It is word for word of the proof of Theorem 6.6 in [18]. \Box

In our analysis below we need also the following auxiliary assertion which is proved in [8].

Theorem 4.10. Let $S \in C^{2,\kappa}$ and $0 < \sigma < \kappa \leq 1$. The null spaces of the singular integral operators

$$2^{-1}I_7 + \mathcal{K}^* : [C^{0,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7, 2^{-1}I_7 + \mathcal{N}^* : [C^{0,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7,$$

are trivial, while the null spaces of the singular integral operators

$$-2^{-1}I_7 + \mathcal{K}^* : [C^{0,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7, -2^{-1}I_7 + \mathcal{N}^* : [C^{0,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7,$$

have the dimension equal to 7. Moreover, the vectors

$$\Psi^{(1)}(x) = (0, -x_3, x_2, 1, 0, 0, 0)^{\top}, \quad \Psi^{(2)}(x) = (x_3, 0, -x_1, 0, 1, 0, 0)^{\top},
\Psi^{(3)}(x) = (-x_2, x_1, 0, 0, 0, 1, 0)^{\top}, \quad \Psi^{(4)}(x) = (1, 0, 0, 0, 0, 0, 0)^{\top},
\Psi^{(5)}(x) = (0, 1, 0, 0, 0, 0, 0)^{\top}, \quad \Psi^{(6)}(x) = (0, 0, 1, 0, 0, 0, 0)^{\top},
\Psi^{(7)}(x) = (0, 0, 0, 0, 0, 0, 1)^{\top},$$
(4.24)

restricted onto the surface S, $\{\Psi^{(k)}(x), x \in S\}_{k=1}^{k=7}$, represent a basis of the null space of the operator $[-2^{-1}I_7 + \mathcal{N}^*]$, while the system of vectors $\{g^{(k)}(x), x \in S\}_{k=1}^{k=7}$ with

$$g^{(k)} = [\mathcal{H}^*]^{-1} \Psi^{(k)}, \qquad k = \overline{1, 7},$$

represents a basis of the null space of the operator $[-2^{-1}I_7 + \mathcal{K}^*]$.

5. Reduction to integral equations and existence theorems

We look for a solution to the interior Neumann type boundary value problem in the form of the single layer potential

$$U(x) = V(g)(x) = \int_{S} \Gamma(x - y) g(y) dS_y, \qquad x \in \Omega^+,$$
(5.1)

where $g \in [C^{0,\sigma}(S)]^7$ is an unknown density vector-function. Evidently, the vector-function (5.1) automatically satisfies the differential equation (3.1), while the boundary condition (3.2) leads to the following singular integral equation

$$-2^{-1}g(x) + \mathcal{K}g(x) = F(x), \qquad x \in S, \tag{5.2}$$

where the operator \mathcal{K} is defined in Theorem 4.4. Due to Theorem 4.5 the operator $[-2^{-1}I_7 + \mathcal{K}]$ is an elliptic singular integral operator of normal type, i.e., its symbol matrix is non-degenerate and for the equation (5.2) the Fredholm theorems hold.

To analyse the solvability of equation (5.2) we need to investigate the null spaces of the operator $[-2^{-1}I_7 + \mathcal{K}]$ and its adjoint one.

$$-2^{-1}g(x) + \mathcal{K}g(x) = 0, \qquad x \in S.$$
(5.3)

In what follows we show that (5.3) possesses only seven independent solutions, i.e.,

$$\dim \ker[-2^{-1}I_7 + \mathcal{K}] = 7.$$

Indeed, let $g_0 \in \ker[-2^{-1}I_7 + \mathcal{K}]$ and consider the single layer potential $V(g_0)$. It is evident that $V(g_0)$ solves the homogeneous Neumann type interior boundary value problem (3.1)–(3.2) with F = 0. Therefore in view of Remark 3.2, the following representation

$$V(g_0)(x) = \sum_{k=1}^{7} C_k \Phi^{(k)}(x), \qquad x \in \Omega^+,$$
(5.4)

holds with appropriately chosen constants C_k . Here the vector-functions $\Phi^{(k)}$, $k = \overline{1,7}$, are defined in (3.8). Theorem and the relation (5.4) imply

$$\{V(g_0)(x)\}^+ = \mathcal{H}(g_0)(x) = \sum_{k=1}^7 C_k \Phi^{(k)}(x), \quad x \in S,$$

where the integral operator \mathcal{H} is defined in Theorem 4.4. By the invertibility of the operator \mathcal{H} (see Remark 4.6), we deduce

$$g_0(x) = \sum_{k=1}^{7} C_k \mathcal{H}^{-1} \Phi^{(k)}(x), \qquad x \in S.$$

Further, since the system $\{\Phi^{(k)}(x)\}_{k=1}^7$ is linearly independent in Ω^+ , the same system is linearly independent on S as well. Indeed, if there are constants $b_k, k = \overline{1,7}$, such that $\sum_{k=1}^7 |b_k| \neq 0$ and

$$\sum_{k=1}^{7} b_k \Phi^{(k)}(x) = 0, \qquad x \in S,$$

then it follows that the vector-function

equation

$$U(x) := \sum_{k=1}^{7} b_k \Phi^{(k)}(x), \qquad x \in \Omega^+,$$

solves the interior Dirichlet type problem in Ω^+ and due to the uniqueness Theorem 2.2 in [7], we conclude $U(x) = 0, x \in \Omega^+$, which contradicts to the linear independency of the system $\{\Phi^{(k)}(x)\}_{k=1}^7$ in Ω^+ .

Let us now prove that the system

$$\{\mathcal{H}^{(-1)}\Phi^{(k)}(x)\}_{k=1}^7, \qquad x \in S,$$

is also linearly independent. Indeed, let there be constants d_k , $k = \overline{1,7}$, such that $\sum_{k=1}^{7} |d_k| \neq 0$ and

$$\sum_{k=1}^{7} d_k \mathcal{H}^{-1} \Phi^{(k)}(x) = 0, \qquad x \in S.$$

Applying the operator \mathcal{H} to this equation we get

$$\sum_{k=1}^{7} d_k \, \Phi^{(k)}(x) = 0, \qquad x \in S,$$

which contradicts the linear independency of the system $\{\Phi^{(k)}(x)\}_{k=1}^7$ on S.

Further, let us introduce the notation

$$g^{(k)}(x) := \mathcal{H}^{-1}\Phi^{(k)}(x), \qquad x \in S.$$
 (5.5)

It is evident that the system $\{g^{(k)}(x)\}_{k=1}^7$ is linearly independent, implying that

dim ker
$$[-2^{-1}I_7 + \mathcal{K}] \ge 7$$
.

On the other hand, from the above arguments it follows that the system $\{g^{(k)}(x)\}_{k=1}^7$ is a basis of the null space ker $[-2^{-1}I_7 + \mathcal{K}]$, i.e., any solution to the homogeneous equation (5.3) is representable in the form

$$g_0 = \sum_{k=1}^{7} C_k g^{(k)}(x), \qquad x \in S$$

with some constants C_k . Thus we have proven the following assertion.

Theorem 5.1. Let $S \in C^{2,\alpha}$ with $0 < \alpha \leq 1$. The dimension of the null space of the singular integral operator $[-2^{-1}I_7 + \mathcal{K}]$ equals to seven and the system $\{\mathcal{H}^{-1}\Phi^{(k)}(x)\}_{k=1}^7$, $x \in S$, is its basis, where $\Phi^{(k)}$, $k = \overline{1,7}$, are given in (3.8). Moreover, if the nonhomogeneous equation (5.2) is solvable and g^* is its particular solution, then the vector

$$g = g^* + \sum_{k=1}^{7} C_k g^{(k)}$$

with $g^{(k)}$ given by (5.5) and C_k being arbitrary constants, solves the same nonhomogeneous equation.

To derive the necessary and sufficient conditions for the nonhomogeneous equation (5.2) to be solvable, we need to analyze the null space of the corresponding adjoint operator $[-2^{-1}I_7 + \tilde{\mathcal{K}}]$, where $\tilde{\mathcal{K}}$ is the operator adjoint to \mathcal{K} in the sense of the space $[L_2(S)]^7$, i.e., $(\mathcal{K}g, \varphi)_{[L_2(S)]^7} = (g, \tilde{\mathcal{K}}\varphi)_{[L_2(S)]^7}$ for all $g, \varphi \in [L_2(S)]^7$.

From the following chain of equalities

$$(\mathcal{K}g,\varphi)_{[L_2(S)]^7} = \int_S \Big(\int_S \mathcal{P}\big(\partial_x, n(x)\big)\Gamma(x-y)g(y)dS_y\Big)\varphi(x)dS_x$$

$$= \int_{S} \left(\int_{S} \mathcal{P}(\partial_{x}, n(x)) \Gamma(x - y) g(y) \varphi(x) dS_{x} \right) dS_{y}$$

$$= \int_{S} \left(\int_{S} g(y) \left[\mathcal{P}(\partial_{x}, n(x)) \Gamma(x - y) \right]^{\top} \varphi(x) dS_{x} \right) dS_{y}$$

$$= \int_{S} g(y) \left(\int_{S} \left[\mathcal{P}(\partial_{x}, n(x)) \Gamma(x - y) \right]^{\top} \varphi(x) dS_{x} \right) dS_{y}$$

$$= \int_{S} g(x) \left(\int_{S} \left[\mathcal{P}(\partial_{y}, n(y)) \Gamma(y - x) \right]^{\top} \varphi(y) dS_{y} \right) dS_{x}$$

and taking into account that $\Gamma(y-x) = [\Gamma^*(x-y)]^{\top}$, we get

$$\widetilde{\mathcal{K}}\,\varphi(x) = \int_{S} \left[\mathcal{P}\big(\partial_{y}, n(y)\big) \big(\Gamma^{*}(x-y)\big)^{\top} \right]^{\top} \varphi(y) dS_{y}, \qquad x \in S,$$

whence it follows that the operator $\widetilde{\mathcal{K}}$ coincides with the operator \mathcal{N}^* defined in (4.22), i.e., $\mathcal{N}^* = \widetilde{\mathcal{K}}$. Therefore the following assertion immediately follows from Theorem 4.10.

Theorem 5.2. Let $S \in C^{2,\alpha}$ with $0 < \alpha \leq 1$. The null space of the operator $[-2^{-1}I_7 + \tilde{\mathcal{K}}]$ is seven dimensional and the system of vector-functions $\{\Psi^{(k)}(x)\}_{k=1}^7$, $x \in S$, with $\Psi^{(k)}$, $k = \overline{1,7}$ defined in (4.24), represents its basis.

Now we are in the position to formulate the main existence results which directly follow from Theorems 5.1 and 5.2.

Theorem 5.3. Let $S \in C^{2,\alpha}$ and $F \in C^{0,\sigma}(S)$ with $0 < \sigma < \alpha \leq 1$. For solvability of the nonhomogeneous equation (5.2) the necessary and sufficient conditions read as follows

$$(F, \Psi^{(k)})_{[L_2(S)]^7} \equiv \int_S F(x) \cdot \Psi^{(k)}(x) \, dS = 0, \qquad k = \overline{1, 7}, \tag{5.6}$$

where the system of vector-functions $\{\Psi^{(k)}(x)\}_{k=1}^7$, $x \in S$, is defined in (4.24).

Proof. It immediately follows from the general theory of singular integral equations (see, e.g., [6. Ch. IV], [11]). since the operator $[-2^{-1}I_7 + \mathcal{K}]$ is of normal type with index equal to zero and the system of vector-functions $\{\Psi^{(k)}(x)\}_{k=1}^7, x \in S$, defined in (4.24) represents the basis of the null space of the adjoint operator $[-2^{-1}I_7 + \mathcal{K}]$. Therefore for a given right hand side vector-function F the nonhomogeneous equation (5.2) is solvable if and only if the orthogonality conditions (5.6) are satisfied.

Theorem 5.4. Let $S \in C^{2,\alpha}$ and $F \in C^{0,\sigma}(S)$ with $0 < \sigma < \alpha \leq 1$. The nonhomogeneous Neumann type boundary value problem (3.1)–(3.2) is solvable if and only if the boundary vector-function F satisfies the orthogonality conditions (5.6).

Moreover, a solution U to the interior Neumann type boundary value problem is representable by the single layer potential (5.1), where the density vector-function g is defined by the singular integral equation (5.2). The solution vector U is defined modulo a linear combination

$$U^{(*)}(x) = \sum_{k=1}^{7} C_k \Phi^{(k)}(x), \qquad x \in \Omega^+,$$

where C_k are arbitrary constants and $\Phi^{(k)}$, $k = \overline{1,7}$, are defined in (3.8).

Proof. It directly follows from Theorems 5.1, 5.2, and 5.3.

6. Appendix

6.1 Particular solutions the problem (3.4)

Unlike the classical thermoelasticity theory, explicit construction of a particular solution $\widetilde{V}_0 = (u_0, \omega_0)^{\top}$ of the problem (3.4) in Ω^+ is problematic. If the condition

$$\frac{\eta}{2\mu + 3\lambda} = \frac{\zeta}{2\chi + 3\delta}$$

is satisfied, then for an arbitrary domain Ω^+ a particular solution to the problem (3.4) reads as

$$\widetilde{V}_0 = \frac{\eta}{2\mu + 3\lambda} (x, 0)^{\top} = \frac{\eta}{2\mu + 3\lambda} (x_1, x_2, x_3, 0, 0, 0)^{\top}.$$

If the domain Ω^+ is a sphere B(0, R) centered at the origin and radius R, then a particular solution $\widetilde{V}_0 = (u_0, \omega_0)^{\top}$ to the problem (3.4) can be constructed without any restriction of material parameters and reads as follows [18]

$$u_0(x) = A_1 x^{\top} - A_2(\delta + 2\chi) \frac{dg_0(r)}{dr} \widetilde{n}(x), \qquad \omega_0(x) = A_2(\lambda + 2\mu) \frac{dg_0(r)}{dr} \widetilde{n}(x),$$

where

$$\begin{aligned} x &= (x_1, x_2, x_3), \quad r = |x|, \quad \tilde{n}(x) = \frac{x^{\top}}{r}, \quad g_0(r) = \frac{J_{1/2}(i\lambda_1 r)}{\sqrt{r}}, \quad \lambda_1 \lambda^2 = \frac{4\alpha(\lambda + 2\mu)}{d_2}, \\ A_1 &= \frac{4\eta}{RD} \Big\{ \Big[\chi(\delta + 2\chi) - \gamma(\lambda + 2\mu) \Big] \frac{dg_0(R)}{dR} + \alpha(\lambda + 2\mu)Rg_0(R) \Big\} - \frac{4\zeta(\mu\delta - \lambda\chi)}{RD} \frac{dg_0(R)}{dR}, \\ A_2 &= \frac{\zeta(3\lambda + 2\mu) - \eta(3\delta + 2\chi)}{D}, \\ D &= \Big\{ (3\lambda + 2\mu) \Big[\chi(\delta + 2\chi) - \gamma(\lambda + 2\mu) \Big] + (3\delta + 2\chi)(\lambda\chi - \mu\delta) \Big\} \frac{4}{R} \frac{dg_0(R)}{dR} \\ &+ 4\alpha(\lambda + 2\mu)(3\lambda + 2\mu)g_0(R), \\ d_2 &:= (\lambda + 2\mu)(\beta + 2\gamma) - (\delta + 2\chi)^2 > 0. \end{aligned}$$

Here $J_{1/2}(i\lambda_1 r)$ is the Bessel function of the first order. Note that the vector $\tilde{n}(x)$ for $x \in \partial B(0, R)$ coincides with the exterior normal vector at the point $x \in \partial B(0, R)$.

6.2 Fundamental solution

The fundamental matrix of the operator of elastostatics $L(\partial)$, which solves the distributional matrix differential equation $L(\partial_x)\Gamma(x-y) = I_7 \,\delta(x-y)$ with Dirac's delta distribution $\delta(x-y)$, reads as (for details see [18], [5])

$$\Gamma(x) = \begin{bmatrix} [\Gamma_{pq}^{(1)}(x)]_{3\times3} & [\Gamma_{pq}^{(2)}(x)]_{3\times3} & [\Gamma_{pq}^{(5)}(x)]_{3\times1} \\ [\Gamma_{pq}^{(3)}(x)]_{3\times3} & [\Gamma_{pq}^{(4)}(x)]_{3\times3} & [\Gamma_{pq}^{(6)}(x)]_{3\times1} \\ [\Gamma_{pq}^{(7)}(x)]_{1\times3} & [\Gamma_{pq}^{(8)}(x)]_{1\times3} & \Gamma^{(9)}(x) \end{bmatrix}_{7\times7}$$

$$\begin{split} &= \frac{1}{4\pi} \begin{bmatrix} \widetilde{\Psi}_{1}(x)I_{3} & \widetilde{\Psi}_{2}(x)I_{3} & [0]_{3\times 1} \\ \widetilde{\Psi}_{3}(x)I_{3} & \widetilde{\Psi}_{4}(x)I_{3} & [0]_{3\times 1} \\ [0]_{1\times 3} & [0]_{1\times 3} & \widetilde{\Psi}_{5}(x) \end{bmatrix}_{7\times 7} \\ &- \frac{1}{4\pi} \begin{bmatrix} Q(\partial)\widetilde{\Psi}_{6}(x) & Q(\partial)\widetilde{\Psi}_{7}(x) & [0]_{3\times 1} \\ Q(\partial)\widetilde{\Psi}_{8}(x) & Q(\partial)\widetilde{\Psi}_{9}(x) & [0]_{3\times 1} \\ [0]_{1\times 3} & [0]_{1\times 3} & 0 \end{bmatrix}_{7\times 7} \\ &+ \frac{1}{4\pi} \begin{bmatrix} R(\partial)\Psi_{10}(x) & R(\partial)\Psi_{11}(x) & \nabla^{\top}\Psi_{14}(x) \\ R(\partial)\Psi_{12}(x) & R(\partial)\Psi_{13}(x) & \nabla^{\top}\Psi_{15}(x) \\ [0]_{1\times 3} & [0]_{1\times 3} & 0 \end{bmatrix}_{7\times 7}, \end{split}$$

where

$$\begin{split} \Psi_{1}(x) &= -\frac{\gamma + \varepsilon}{d_{1}|x|} - \frac{1}{d_{1}^{2}(\lambda_{2}^{2} - \lambda_{3}^{2})} \sum_{j=2}^{3} (-1)^{j} \Big\{ 4[\alpha d_{1} + \alpha \mu(\gamma + \varepsilon) + 4\nu(\alpha \chi - \mu \nu)] \\ &+ d_{1}(\gamma + \varepsilon)\lambda_{1}^{2} + \frac{16\alpha^{2}\mu}{\lambda_{j}^{2}} \Big\} \frac{e^{i\lambda_{j}|x|} - 1}{|x|} , \\ \Psi_{2}(x) &= \Psi_{3}(x) = \frac{\chi + \nu}{d_{1}|x|} + \frac{1}{d_{1}^{2}(\lambda_{2}^{2} - \lambda_{3}^{2})} \sum_{j=2}^{3} (-1)^{j} \Big\{ 4\alpha[\mu(\chi + \nu) + 2(\alpha \chi - \mu \nu)] \\ &+ d_{1}(\chi + \nu)\lambda_{j}^{2} \Big\} \frac{e^{i\lambda_{j}|x|} - 1}{|x|} , \\ \Psi_{4}(x) &= -\frac{\mu + \alpha}{d_{1}|x|} - \frac{\mu + \alpha}{d_{1}^{2}(\lambda_{2}^{2} - \lambda_{3}^{2})} \sum_{j=2}^{3} (-1)^{j} (d_{1}\lambda_{j}^{2} + 4\alpha\mu) \frac{e^{i\lambda_{j}|x|} - 1}{|x|} , \\ \Psi_{5}(x) &= -\frac{1}{\kappa'|x|} , \\ \Psi_{5}(x) &= -\frac{1}{\kappa'|x|} , \\ \Psi_{6}(x) &= -\frac{(\lambda + \mu)|x|}{2\mu(\lambda + 2\mu)} + \frac{(\delta + 2\chi)^{2}d_{2}}{4\alpha(\lambda + 2\mu)^{2}} \frac{e^{-\lambda_{1}|x|} - 1}{|x|} + \frac{1}{\lambda_{2}^{2} - \lambda_{3}^{2}} \sum_{j=2}^{3} (-1)^{j} \Big\{ \frac{\gamma + \varepsilon}{d_{1}} \\ &+ \frac{4}{d_{1}^{2}\lambda_{j}^{2}} [\alpha d_{1} + \alpha\mu(\gamma + \varepsilon) + 4\nu(\alpha\chi - \mu\nu)] + \frac{16\alpha^{2}\mu}{d_{1}^{2}\lambda_{j}^{4}} \Big\} \frac{e^{i\lambda_{j}|x|} - 1}{|x|} , \\ \Psi_{7}(x) &= \Psi_{8}(x) = -\frac{\delta + 2\chi}{4\alpha(\lambda + 2\mu)} \frac{e^{-\lambda_{1}|x|} - 1}{|x|} - \frac{1}{\lambda_{2}^{2} - \lambda_{3}^{2}} \sum_{j=2}^{3} (-1)^{j} \Big\{ \frac{\chi + \nu}{d_{1}} \Big\} \end{split}$$

$$\begin{split} &+ \frac{4\alpha}{d_1^2 \lambda_j^2} \Big[\mu(\chi + \nu) + 2(\alpha \chi - \mu \nu) \Big] \Big\} \frac{e^{i\lambda_j |x|} - 1}{|x|}, \\ &\Psi_9(x) = \frac{1}{4\alpha} \frac{e^{-\lambda_1 |x|} - 1}{|x|} + \frac{1}{\lambda_2^2 - \lambda_3^2} \sum_{j=2}^3 (-1)^j \frac{\mu + \alpha}{d_1^2} \Big(d_1 + \frac{4\alpha \mu}{\lambda_j^2} \Big) \frac{e^{i\lambda_j |x|} - 1}{|x|}, \\ &\Psi_{10}(x) = \frac{4}{d_1^2 (\lambda_2^2 - \lambda_3^2)} \sum_{j=2}^3 (-1)^j \Big[\nu d_1 + (\gamma + \varepsilon)(\alpha \chi - \mu \nu) + \frac{4\alpha^2 \chi}{\lambda_j^2} \Big] \frac{e^{i\lambda_j |x|} - 1}{|x|}, \\ &\Psi_{11}(x) = \Psi_{12}(x) = \frac{2}{d_1^2 (\lambda_2^2 - \lambda_3^2)} \sum_{j=2}^3 (-1)^j \Big[2(\chi + \nu)(\mu \nu - \alpha \chi) - \alpha d_1 \\ &- \frac{4\alpha^2 \mu}{\lambda_j^2} \Big] \frac{e^{i\lambda_j |x|} - 1}{|x|}, \\ &\Psi_{13}(x) = \frac{4(\mu + \alpha)(\alpha \chi - \mu \nu)}{d_1^2 (\lambda_2^2 - \lambda_3^2)} \frac{e^{i\lambda_2 |x|} - e^{i\lambda_3 |x|}}{|x|}, \\ &\Psi_{14}(x) = \frac{1}{\kappa'} \Big\{ -\frac{\eta |x|}{2(\lambda + 2\mu)} + \big[\zeta(\lambda + 2\mu) - \eta(\delta + 2\chi) \big] \frac{\delta + 2\chi}{4\alpha(\lambda + 2\mu)^2} \frac{e^{-\lambda_1 |x|} - 1}{|x|} \Big\}, \\ &\Psi_{15}(x) = \frac{\eta(\delta + 2\chi) - \zeta(\lambda + 2\mu)}{4\kappa' \alpha(\lambda + 2\mu)} \frac{e^{-\lambda_1 |x|} - 1}{|x|}; \end{split}$$

where

$$d_{1} := (\mu + \alpha)(\gamma + \varepsilon) - (\varkappa + \nu)^{2}, \quad d_{2} := (\lambda + 2\mu)(\beta + 2\gamma) - (\delta + 2\varkappa)^{2},$$

$$d_{3} := (\mu + \alpha)(\mathcal{I}\sigma^{2} - 4\alpha) + (\gamma + \varepsilon)\varrho\sigma^{2} + 4\alpha^{2}, \quad \lambda_{1}^{2} = \frac{4\alpha(\lambda + 2\mu)}{d_{2}} > 0,$$

$$\lambda_{2,3}^{2} = \frac{4}{d_{1}^{2}} \Big\{ 2(\mu\nu - \alpha\chi)^{2} - \alpha\mu d_{1} \pm i2(\mu\nu - \alpha\chi)\sqrt{(\mu + \alpha[\alpha(\mu\gamma - \chi^{2}) + \mu(\alpha\varepsilon - \nu^{2})])} \Big\}$$

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Authors' address:

D. Ivanidze and D. Natroshvili
Department of Mathematics
Georgian Technical University
77, M. Kostava St., Tbilisi 0175, Georgia
E-mail: diana.ivanize@gmail.com
natrosh@hotmail.com