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WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR ONE CLASS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS TAKING INTO ACCOUNT DELAY PERTURBATION

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Abstract. In the present paper, for the quasilinear functional differential equation with the discontinuous initial condition we formulate the theorems on the continuous dependence of the solution, on perturbations of the initial moment, the variable delay entering in the phase coordinates, the initial vector, the initial functions and the nonlinear term of right-hand side. The discontinuous initial condition means that the values of the initial function and trajectory, generally, do not coincide at the initial moment.

Keywords and phrases: Neutral functional differential equation; well-posedness of the Cauchy problem; discontinuous initial condition.

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Let $\mathbb{R}^n_x$ be the n-dimensional vector space of points $x = (x^1, ..., x^n)^T$, where T means transpose; let $I = [a, b] \subset \mathbb{R}^1_t$ be a finite interval, let $O \subset \mathbb{R}^n_x$ be a open set; let D be the set of continuously differentiable functions $\tau(t)$ satisfying the conditions: $\tau(t) < t,\ \dot{\tau}(t) > 0$ with

$$\inf\{\tau(a) : \tau \in D\} = \hat{\tau} < \infty, ||\tau|| = \sup\{|\tau(t)| : t \in I\}.$$

Let E_φ be the space of piecewise-continuous functions $\varphi : I_1 = [\hat{\tau}, b] \to \mathbb{R}^n_x$, with finitely many discontinuity points of the first kind, $||\varphi|| = \sup\{|\varphi(t)| : t \in I_1\}$; let $\Phi_1 = \{\varphi \in E_\varphi : \varphi(t) \in O, t \in I_1\}$ be the set of initial functions with $cl\varphi(I_1) \subset O$; let Φ_2 be the set of bounded measurable functions $h : I_1 \to \mathbb{R}^n_x,\ ||h|| = \sup\{|h(t)| : t \in I_1\}$.

Let E_f be the space of functions $f : I \times O^2 \to \mathbb{R}^n_x$ satisfying the following conditions: the function $f(\cdot, x, y) : I \to \mathbb{R}^n_x$ is measurable for each fixed $(x, y) \in O^2$; for an arbitrary compact set $K \subset O$ and for $f \in E_f$ there exist functions $m_{f,K}(\cdot),\ L_{f,K}(\cdot) \in L(I, [0, \infty))$, such that for almost all $t \in I$ the following inequalities are fulfilled

$$| f(t, x, y) |\le m_{f,K}(t), \quad \forall (x, y) \in K^2,$$

$$| f(t, x_1, y_1) - f(t, x_2, y_2) |\le L_{f,K}(t)(| x_1 - x_2 | + | y_1 - y_2 |),$$
$$\forall (x_i, y_i) \in K^2,\ \ i = 1, 2.$$

To each element $\mu = (t_0, \tau, x_0, \varphi, h, f) \in \Lambda = I \times D \times O \times \Phi_1 \times \Phi_2 \times E_f$ we put in correspondence the quasilinear neutral functional differential equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f(t, x(t), x(\tau(t))) \qquad (1)$$

with the initial condition

$$x(t)=\varphi(t),\ \dot{x}(t)=h(t),\ t\in[\hat{\tau},t_0),\ \ x(t_0)=x_0. \qquad (2)$$

Here $A(t)$ is a given continuous matrix function with dimension $n\times n; \sigma\in D$ is a fixed function.

The condition (2) is said to be the discontinuous initial condition since generally $x(t_0)\neq\varphi(t_0)$.

Definition 1. Let $\mu=(t_0,\tau,x_0,\varphi,h,f)\in\Lambda,\ t_0\in[a,b)$. A function $x(t)=x(t;\mu)\in O,\ t\in[\hat{\tau},t_1],\ t_1\in(t_0,b]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to element μ and defined on the interval $[\hat{\tau},t_1]$ if it satisfies condition (2) and it is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) almost everywhere on $[t_0,t_1]$.

If t_1-t_0 is a sufficiently small number, then the unique solution always corresponds to μ.

To formulate the main results, we introduce the following sets:

$$W(K,\alpha_1)=\Big\{\delta f\in E_f:\exists m_{\delta f,K},L_{\delta f,K}\in L(I,[0,\infty)),$$
$$\int_I[m_{\delta f,K}(t)+L_{\delta f,K}(t)]dt\le\alpha_1\Big\},$$

where $K\subset O$ is a compact set and $\alpha_1>0$ is a given number independent of δf;

$$V_{K,\delta}=\Big\{\delta f\in E_f:\Big|\int_{s_1}^{s_2}\delta f(t,x,y)dt\Big|\le\delta,\ \forall(s_1,s_2,x,y)\in I^2\times K^2\Big\},$$

$$B(t_{00};\delta)=\{t_0\in I:|t_0-t_{00}|<\delta\},\ \ B(x_{00};\delta)=\{x_0\in O:|x_0-x_{00}|<\delta\},$$
$$V(\tau_0;\delta)=\{\tau\in D:||\tau-\tau_0||<\delta\},V(\varphi_0;\delta)=\{\varphi\in\Phi_1:||\varphi-\varphi_0||<\delta\},$$
$$V(h_0;\delta)=\{h\in\Phi_2:||h-h_0||<\delta\},$$

where $t_{00}\in I,\ x_{00}\in O$ are fixed points; $\tau_0\in D,\ \varphi_0\in\Phi_1,\ h_0\in\Phi_2$ are fixed functions.

Theorem 1. *Let $x_0(t)=x(t;\mu_0)$, where $\mu_0=(t_{00},\tau_0,x_{00},\varphi_0,h_0,f_0)\in\Lambda$, is the solution defined on $[\hat{\tau},t_{10}]$, $t_{10}<b$; let $K_1\subset O$ be a compact set containing a certain neighborhood of the set $cl\varphi_0(I_1)\cup x_0([t_{00},t_{10,}])$. Then the following assertions hold:*

1. *there exist numbers $\delta_i>0,\ i=0,1$, such that, to each element*

$$\mu\in V(\mu_0;K_1,\delta_0,\alpha_1)=B(t_{00};\delta_0)\times V(\tau_0;\delta_0)\times B(x_{00};\delta_0)\times V(\varphi_0;\delta_0)$$
$$\times V(h_0;\delta_0)\times\Big[f_0+W(K_1,\alpha_1)\cap V_{K_1,\delta_0}\Big]$$

we put in correspondence the solution $x(t;\mu)$ defined on the interval $[\hat{\tau},t_{10}+\delta_1]\subset I_1$ and satisfying the condition $x(t;\mu)\in intK_1,\ t\in[\hat{\tau},t_{10}+\delta_1]$;

2. *for an arbitrary $\varepsilon>0$ there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0]$ such that for any $\mu\in V(\mu_0;K_1,\delta_2,\alpha_1)$ the following inequality holds:*

$$|x(t;\mu)-x(t;\mu_0)|\le\varepsilon,\ \ \forall t\in[s_1,t_{10}+\delta_1],\ \ s_1=\max\{t_{00},t_0\};$$

3. *for an arbitrary* $\varepsilon > 0$ *there exists a number* $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0]$ *such that for any* $\mu \in V(\mu_0; K_1, \delta_3, \alpha_1)$ *the following inequality holds:*

$$\int_{\hat{\tau}}^{t_{10}+\delta_1} |x(t;\mu) - x(t;\mu_0)|dt \leq \varepsilon.$$

In the space $E_\mu - \mu_0$, where $E_\mu = \mathbb{R}^1_t \times D \times \mathbb{R}^n_x \times \Phi_1 \times \Phi_2 \times E_f$ introduce the set of variation:

$$\Im = \Big\{ \delta\mu = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta h, \delta f) \in E_\mu - \mu_0 : |\, \delta t_0 \,| \leq \alpha_2, |\, \delta\tau \,| \leq \alpha_2,$$

$$|\, \delta x_0 \,| \leq \alpha_2, \|\delta\varphi\|_1 \leq \alpha_2, \|\delta h\|_1 \leq \alpha_2, \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i,\ |\lambda_i| \leq \alpha_2,\ i = \overline{1,k} \Big\},$$

where $\alpha_2 > 0$ is a fixed number, $\delta f_i \in E_f,\ i = \overline{1,k}$, are fixed functions.

The following theorem is a simple consequence of theorem 1.

Theorem 2. *Let* $x_0(t) = x(t;\mu_0)$ *be the solution defined on* $[\hat{\tau}, t_{10}],\ t_{i0} \in (a,b),\ i = 0,1$; *let* $K_1 \subset O$ *be a compact set containing a certain neighborhood of the set* $cl\varphi_0(I_1) \cup x_0([t_{00}, t_{10,}])$. *Then the following assertions hold:*

4. *there exist numbers* $\varepsilon_1 > 0,\ \delta_1 > 0$, *such that, for an arbitrary* $(\varepsilon, \mu) \in [0, \varepsilon_1] \times \Im$ *the element* $\mu_0 + \varepsilon\delta\mu \in \Lambda$, *we put in correspondence the solution* $x(t; \mu_0 + \varepsilon\delta\mu)$ *defined on the interval* $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ *and satisfying the condition* $x(t; \mu_0 + \varepsilon\delta\mu) \in intK_1,\ t \in [\hat{\tau}, t_{10} + \delta_1]$;

5. *the following relations hold:*

$$\lim_{\varepsilon \to 0} \sup\{|x(t; \mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)| : t \in [s_1, t_{10} + \delta_1]\} = 0, \quad s_1 = \max\{t_{00}, t_{00} + \varepsilon\delta t_0\};$$

$$\lim_{\varepsilon \to 0} \int_{\hat{\tau}}^{t_{10}+\delta_1} |x(t; \mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)|dt = 0$$

uniformly for $\delta\mu \in \Im$.

Now let us formulate the theorem on the continuous dependence of the solution for an equation whose righthand side depends on the control. Let $U_0 \subset \mathbb{R}^r_u$ be an open set and let Ω be the set of measurable functions $u(t) \in U_0,\ t \in I$, satisfying the condition: $clu(I)$ is a compact set in $\mathbb{R}^r_u$ and $clu(I) \subset U_0$.

To each element $\rho = (t_0, \tau, x_0, \varphi, h, u) \in \Lambda_1 = [a,b) \times D \times O \times \Phi_1 \times \Phi_2 \times \Omega$ we assign the control neutral functional differential equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + g(t, x(t), x(\tau(t)), u(t)) \tag{3}$$

with the initial condition (2). Here the function $g(t,x,y,u)$ is defined on $I \times O^2 \times U_0$ and satisfies the following conditions: for each fixed $(x,y,u) \in O^2 \times U_0$ the function $g(\cdot, x, y, u) : I \to \mathbb{R}^n_u$ is measurable; for each compact sets $K \subset O$ and $U \subset U_0$ there exist functions $m_{K,U}, L_{K,U} \in L(I, [0,\infty))$ such that for almost all $t \in I$

$$|g(t,x,y,u)| \leq m_{K,U}(t), \quad \forall (x,y,u) \in K^2 \times U,$$

$$|g(t,x_1,y_1,u_1)-g(t,x_2,y_2,u_2)|\le L_{K,U}(t)\Big[|x_1-x_2|+|y_1-y_2|+|u_1-u_2|\Big],$$

$$\forall(x_1,x_2,y_1,y_2,u_1,u_2,)\in K^4\times U^2.$$

Definition 2. Let $\rho=(t_0,\tau,x_0,\varphi,h,u)\in\Lambda_1$. A function $x(t)=x(t;\rho)\in O,\ t\in[\hat\tau,t_1],\ t_1\in(t_0,b]$, is called a solution of equation (3) with the initial condition (2) or a solution corresponding to element ρ and defined on the interval $[\hat\tau,t_1]$, if it satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (3) almost everywhere on $[t_0,t_1]$.

Theorem 3. *Let $x_0(t)=x(t;\rho_0)$, where $\rho_0=(t_{00},\tau_0,x_{00},\varphi_0,h_0,u_0)\in\Lambda_1$, be a solution defined on $[\hat\tau,t_{10}]$, $t_{10}<b$; let $K_1\subset O$ be a compact set containing a certain neighborhood of the set $cl\varphi_0(I_1)\cup x_0([t_{00},t_{10,}])$. Then the following assertions hold:*

6. *there exist numbers $\delta_i>0,\ i=0,1$, such that, to each element $\rho\in\hat V(\rho_0;\delta_0)=B(t_{00};\delta_0)\times V(\tau_0;\delta_0)\times B(x_{00};\delta_0)\times V(\varphi_0;\delta_0)\times V(h_0;\delta_0)\times V(u_0;\delta_0)$ corresponds the solution $x(t;\rho)$ defined on the interval $[\hat\tau,t_{10}+\delta_1]\subset I_1$ and satisfying the condition $x(t;\rho)\in intK_1$; here $V(u_0;\delta_0)=\{u\in\Omega:||u-u_0||<\delta\}$;*

7. *for an arbitrary $\varepsilon>0$ there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0]$ such that for any $\rho\in\hat V(\rho_0;\delta_0)$ the following inequality holds:*

$$|x(t;\rho)-x(t;\rho_0)|\le\varepsilon,\ \ \forall t\in[s_1,t_{10}+\delta_1],\ \ s_1=\max\{t_{00},t_0\};$$

8. *for an arbitrary $\varepsilon>0$ there exists a number $\delta_3=\delta_3(\varepsilon)\in(0,\delta_0]$ such that for any $\rho\in\hat V(\rho_0;\delta_0)$ the following inequality holds:*

$$\int_{\hat\tau}^{t_{10}+\delta_1}|x(t;\rho)-x(t;\rho_0)|dt\le\varepsilon.$$

Some comments. Theorems analogous to Theorem 1-3, without perturbation of variable delay, for various classes of functional differential equations are proved in [1-3]. In Theorem 1 perturbations of the nonlinear term of right-hand side of equation (1) are small in the integral sense. Theorems 1-3 play an important role in proving necessary optimality conditions and variation formulas of solution [1,4-7].

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