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ON THE WELL-POSEDNESS OF A CLASS OF THE OPTIMAL CONTROL PROBLEM WITH DISTRIBUTED DELAY

Dvalishvili P.

Abstract. A theorem of the well-posedness is given for the linear with respect to control optimal problem, when perturbations of the right-hand side of a differential equation and an integrand are small in the integral sense.

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Let $a < t_{01} < t_{02} < t_{11} < t_{12} < b, \theta > 0, \tau > 0$ be given numbers and let R_x^n be the n-dimensional vector space of points $x = (x^1, ..., x^n)^T$, where T means transpose; suppose that $O \subset R_x^n$ is an open set and $U \subset R_u^r$ is a compact and convex set, the $n \times r$ -dimensional matrix-function $f(t, x)$ is continuous on the set $I \times O$ and continuously differentiable with respect to $x \in O$, where $I = [a, b]$. Further, let the scalar function $f^0(t, x, u)$ be continuous on the set $I \times O \times U$ and convex in $u \in U$; let Φ be the set of continuous initial functions $\varphi(t) \in O, t \in [a - \tau, t_{02}]$; let Ω be the set of measurable control functions $u(t) \in U, t \in [a - \theta, b]$.

To each element

$$w = (t_0, t_1, u(\cdot)) \in W = [t_{01}, t_{02}] \times [t_{11}, t_{12}] \times \Omega$$

we assign the differential equation linear with respect to control

$$\dot{x}(t) = \int_{-\theta}^{0} \Big\{ \int_{-\tau}^{0} f(t, x(t+s))u(t+\xi)ds \Big\} d\xi, t \in [t_0, t_1] \tag{1}$$

with the initial condition

$$x(t) = \varphi_0(t), t \in [t_0 - \tau, t_0), x(t_0) = x_{00}, \tag{2}$$

where $\varphi_0(\cdot) \in \Phi$ is a given initial function, $x_{00} \in O$ is a given initial vector.

Equation (1) is called a differential equation with distributed delay in phase coordinates and in controls.

Definition 1. Let $w = (t_0, t_1, u(\cdot)) \in W$. A function $x(t) = x(t; w) \in O, t \in [t_0 - \tau, t_1]$ is called solution corresponding to the element w, if the conditions (1) and (2) are fulfilled. Moreover, the function $x(t), t \in [t_0, t_1]$ is absolutely continuous and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Definition 2. An element $w = (t_0, t_1, u(\cdot) \in W$ is admissible if there exists the corresponding solution $x(t) = x(t; w), t \in [t_0 - \tau, t_1]$ and the condition

$$x(t_1) = x_1 \tag{3}$$

is fulfilled. Here $x_1 \in O$ is a given point and also $x_1 \neq x_{00}$.

The set of admissible elements will be denoted by W_0.

Definition 3. An element $w_0 = (t_{00}, t_{10}, u_0(\cdot)) \in W_0$ is called optimal, if

$$J_0 = J(w_0) = \inf_{w \in W_0} J(w), \tag{4}$$

where

$$J(w) = \int_{-\theta}^{0} \Big\{ \int_{-\tau}^{0} f^0(t, x(t+s), u(t+\xi))ds \Big\} d\xi, x(t) = x(t; w).$$

Problem (1)-(4) is called an optimal problem with distributed delay. The element w_0 is called the solution of problem (1)-(4).

To formulate the main result we need the following notation: E is the space of vector functions $G(t,x) = (g^0(t,x), g^1(t,x), ..., g^n(t,x))^T$ which satisfy the following conditions: for every $x \in O$ the function $G(t,x)$ is measurable on I; for every $G \in E$ and any compact set $K \subset O$ there exist functions $m_{G,K}(\cdot), L_{G,K}(\cdot) \in L_1(I; R_+), R_+ = [0, \infty)$ such that the inequalities

$$|G(t,x)| \leq m_{G,K}(t), \forall x \in K,$$

$$|G(t,x) - G(t,y)| \leq L_{G,K}(t)|x-y|, \forall (x,y) \in K^2$$

are fulfilled for almost all $t \in I$.

Let $K \subset O$ be a compact set, $C > 0$ is a given number. Denote by W_K the set of perturbations:

$$W_K = \Big\{ G \in E \mid \exists m_{G,K}(\cdot), L_{G,K}(\cdot) \in L_1(I; R_+), \int_I \Big[m_{G,K}(t) + L_{G,K}(t) \Big] dt \leq C \Big\}.$$

Furthermore,

$$V_{\delta,K} = \Big\{ G \in W_K \mid \sup_{(t',t'',x) \in I^2 \times K} \Big| \int_{t'}^{t''} G(s,x)ds \Big| \leq \delta \Big\}, \delta > 0;$$

$$B_{x_{00},\delta} = \Big\{ x_0 \in O \mid |x_0 - x_{00}| \leq \delta \Big\}, B_{\varphi_0,\delta} = \Big\{ \varphi_0(\cdot) \in \Phi | \|\varphi_0 - \varphi\| \leq \delta \Big\},$$

$$\|\varphi_0 - \varphi\| = \max_{t \in [a-\tau, t_{02}]} |\varphi_0(t) - \varphi(t)|.$$

Theorem 1. *Let the following conditions be fulfilled:*

1) $W_0 \neq \emptyset$;

2) *there exists a compact set* $K_0 \in O$ *such that*

$$x(t; w) \in K_0, t \in [t_0 - \tau, t_1], \forall w \in W_0.$$

Then for any $\varepsilon > 0$ *there exists a number* $\delta = \delta(\varepsilon) > 0$ *such that for every*

$$\mu = (x_0, \varphi(\cdot), G) \in B_{x_{00},\delta} \times B_{\varphi_0,\delta} \times V_{\delta,K_1}$$

the perturbed optimal control problem

$$\dot{x}(t)=\int_{-\theta}^{0}\Big\{\int_{-\tau}^{0}\Big[f(t,x(t+s))u(t+\xi)+g(t,x(t+s))\Big]ds\Big\}d\xi, t\in[t_0,t_1],$$

$$x(t)=\varphi(t), t\in[t_0-\tau,t_0), x(t_0)=x_0, x(t_1)\in B_{x_1,\delta},$$

$$J(w;\mu)=\int_{-\theta}^{0}\Big\{\int_{-\tau}^{0}\Big[f^0(t,x(t+s),u(t+\xi))+g^0(t,x(t+s))\Big]ds\Big\}d\xi\to\min$$

has the solution $w_0(\mu)=(t_{00}(\mu),t_{10}(\mu),u_0(\cdot;\mu))$. Also,if

$$\mu_i=(x_{0i},\varphi_i(\cdot),G_i)\in B_{x00,\delta_i}\times B_{\varphi_0,\delta_i}\times V_{\delta_i,K_1}, i=1,2,...,$$

where $\delta_i=\delta(\varepsilon_i),\varepsilon_i\to 0,$ *then*

$$\lim_{i\to\infty}J(w_0(\mu_i);\mu_i)=J_0.$$

Moreover, from the sequence $w_i, i=1,2,...$ *we can choose a subsequence*

$$w_0(\mu_{i_k})=(t_{00}(\mu_{i_k}),t_{10}(\mu_{i_k}),u_0(\cdot;\mu_{i_k})), k=1,2,...$$

such that

$$\lim_{k\to\infty}t_{00}(\mu_{i_k})=t_{00}, \lim_{k\to\infty}t_{10}(\mu_{i_k})=t_{10},$$

$$\lim_{k\to\infty}u_0(t;\mu_{i_k})=u_0(t), \text{ weakly in } L_1([a-\theta,b];U)$$

and $w_0=(t_{00},t_{10},u_0(\cdot))$ *is a solution of the problem (1)-(4). Here* $g=(g^1,...,g^n)^T$, $K_1\subset O$ *is a compact set containing a certain neighborhood of the compact* K_0.

Some comments.

c1. If the problem (1)-(4) has a unique solution $w_0=(t_{00},t_{10},u_0(\cdot))$, then we have

$$\lim_{i\to\infty}t_{00}(\mu_i)=t_{00}, \lim_{i\to\infty}t_{10}(\mu_i)=t_{10},$$

$$\lim_{i\to\infty}u_0(t;\mu_i)=u_0(t), \text{ weakly in } L_1([a-\theta,b];U).$$

c2. A theorem analogous to Theorem 1 also is valid for the following optimal control problem

$$\dot{x}(t)=\int_{-\theta}^{0}\Big\{\int_{-\tau}^{0}\Big[f(t,x(t+s))u(t+\xi)+f_1(t,x(t+s))\Big]ds\Big\}d\xi, t\in[t_0,t_1],$$

$$x(t)=\varphi(t), t\in[t_0-\tau,t_0), x(t_0)=x_{00}, x(t_1)=x_1,$$

$$\int_{-\theta}^{0}\Big\{\int_{-\tau}^{0}\Big[f^0(t,x(t+s),u(t+\xi))+f_1^0(t,x(t+s))\Big]ds\Big\}d\xi\to\min,$$

where $(f_1^0,f_1)^T\in E$ is a given function.

c3. Theorem 1 is proved by the method given in [1].

c4. Theorems of the continuity of the minimum of the integral functional (well-posedness) with respect to perturbations for various classes of optimal control problems, when perturbations are small in the integral sense, are proved in [1-5]. A theorem on the well-posedness for an nonlinear optimal problem with distributed delay in phase coordinates is proved in [6, 7], with distributed delay in phase coordinates and control-in [8, 9].

c5. Finally, we note that various small values are as a rule ignored in the numerical solutions of optimal problems and therefore it is important to establish the connection between initial and perturbed problem.

R E F E R E N C E S

1. Tadumadze T.A. Some problems in the qualitative theory of optimal control. (Russian) *Tbil. Gos. Univ.*, 1983.
2. Tadumadze T.A. Continuity of the minimum of an integral functional in a nonlinear problem of optimal control. (Russian) *Diferentsialnye Uravneniya*, **20**, 6 (1984), 991-995.
3. Tadumadze T.A. Existence of optimal initial data and well-posedness with respect to a functional for a neutral optimal problem. *Seminar of I.N. Vekua institute of Applied Mathematics Proceedings*, **36-37** (2010-2011), 38-41.
4. Kharatishvili G.L., Tadumadze T.A. Regular perturbations in optimal control problems with variable delays and free right end. (Russian) *Dokl. Akad. Nauk. SSSR.* **314**, 1 (1990), 151-155; translation in Soviet Mat. Dokl **42**, 2 (1991), 399-403.
5. Nikolski M.S. Well posedness of an optimal control problem for a linear control system with integral quadratic performance index. (Russian) *Modern Problems of Mathematich. Differential equations, Mathematical Analisis and Their Aplications. Trudy Mat. nst. Steklova*, **166** (1984), 177-185.
6. Dvalishvili F.A. Some problems in the quantitative theory of optimal control with distributed delay. (Russian) *Tbilisi Gos. Univ. Inst. Prikl. Mat. Trudi*, **41** (1991), 83-113.
7. Dvalishvili F.A. On the continuity of the minimum of a functional in a nonlinear optimal control problem with distributed delay. (Russian) *Soobsch. Acad. Nauk Gruzin. SSR* **136**, 2 (1989), 285-288.
8. Dvalishvili P. On well posedness with respect to functional for an optimal control problem with distributed delays. *International workshop on the qualitative theory of differential equations QUALITDE-2011*, Tbilisi 2011 (www.rmi.ge/eng/QUALITDE-2011/workshop-2011.htm).
9. Dvalishvili P., Ramishvili I. A theorem on the continuity of the minimum of an integral functional for one class of optimal problems with distributed delay in controls (to appear).

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Author's address:

P. Dvalishvili
Iv. Javakhishvili Tbilisi State University
Department of Computer Sciences
13, University St., Tbilisi 0186
Georgia
E-mail: pridon.dvalishvili@tsu.ge