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# NONLINEAR MATHEMATICAL MODEL OF DYNAMICS OF VOTERS OF TWO POLITICAL SUBJECTS

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Abstract. In the present paper the nonlinear mathematical model describing dynamics of voters of pro-governmental and oppositional parties (two selective subjects, coalitions) is offered. In model three objects are considered: governmental and administrative structures influencing by means of administrative resources citizens (first of all in opposition adjusted voters) for the purpose of their attraction on the side of pro-governmental party; citizens with the selective voice, at present supporting opposition party; citizens with the selective voice, at present supporting pro-governmental party. In cases constant or variable (in proportion to number of voters of opposition party) uses of administrative resources the problem of Cauchy's for nonlinear system of the differential equations is solved analytically exactly. Conditions for model parameters at which the opposition party (coalition) will win the next elections are found. The mathematical model except theoretical interest has also important practical value, as both sides (the state structures together with pro-governmental party; opposition party) can use results according to the purposes. It allows the sides, according to the chosen strategy, to select parameters of action and to achieve desirable results for themselves.

**Keywords and phrases**: Nonlinear mathematical model, pro-governmental party (coalition), administrative resources, opposition party (coalition), elections.

AMS subject classification (2010): 7M10, 97M70.

### Introduction

Mathematical modeling and computing experiment the last decades gained allround recognition in a science as the new methodology which is roughly developing and widely introduced not only in natural-science and technological spheres, but also in economy, sociology, political science and other public disciplines [1 - 5].

In [6 - 8] the mathematical model of political rivalry devoted to the description of fight occurring in imperious elite competing (but not surely antagonistic) political forces, for example, power branches is considered. It is supposed that each of the parties has ideas of "number" of the power which this party would like to have itself, and about "number" of the power which she would like to have for the partner.

Works [9 - 12] are devoted to creation of mathematical model of such social process what administrative management is. The last can be carried out as at macrolevel (for example, the state) and at microlevel (for example, an educational or research institution, an industrial facility, etc.).

A certain interest represents creation of the mathematical model, allowing to define dynamics of voters of political subjects. It is known that in many countries including developed ones, there are two-party systems. Such systems are characterized by the existence of two largest parties which periodically replace each other in power. And, when in power there is one party, the second is the leading party of opposition. However it doesn't mean that except these two parties in the country there are no other parties, simply their influence on political processes is insignificant. In some countries eventually to change of one of the largest parties can come any else, earlier being in a shadow. For example, in Great Britain in the XIX century and at the beginning of the XX century two largest parties were conservative and liberal. In the XX century Liberal party in this tandem replaced labor, however the two-party system remained. The most rigid option of two-party system exists in the USA. Here only republican and democratic parties apply for the power, other parties almost don't play any role. And in the Congress for it more than two hundred year's history other parties almost were never presented. A version of two-party system is the two-block system. Here not largest parties, and party coalitions appear confronting forces. It is caused by that any party unable to achieve sufficient support of voters independently to create the government therefore parties according to the political orientation and ideological installations unite for increase in the influence. Thus such competing coalitions remain almost in invariable structure throughout quite a long time. Such party system developed, for example, in the Netherlands. Such party systems in which as two main competing forces act, on the one hand, party, and, meet another – the party block also. So, in Australia agrarian and liberal parties make the constant union resisting to the Labour party.

In the real work the nonlinear mathematical model describing dynamics of voters of pro-governmental and oppositional parties (two selective subjects, coalitions) is offered. In the model three objects are considered:

1. The state and administrative structures influencing by means of administrative resources citizens (first of all in opposition adjusted voters) for the purpose of their attraction on the party of pro-governmental party.

2. Citizens with the selective voice, at present supporting opposition party.

3. Citizens with the selective voice, at present supporting pro-governmental party.

## 1. System of the equations and entry conditions

For dynamics description between elections of voters of pro-governmental and oppositional parties (two selective subjects) we offer the following nonlinear mathematical model:

$$\frac{dN_1(t)}{dt} = (\alpha_1(t) - \alpha_2(t))N_1(t)N_2(t) - f(t, N_1(t))$$

$$\frac{dN_2(t)}{dt} = (\alpha_2(t) - \alpha_1(t))N_1(t)N_2(t) + f(t, N_1(t))$$

$$N_1(0) = N_{10}, \quad N_2(0) = N_{20}, \quad N_{10} < N_{20},$$
(1.2)

where  $N_1(t), N_2(t)$  are respectively, a number of the voters supporting oppositional and pro-governmental parties at the moment of time t and  $t \in [0, T], t = 0$  is the moment of time of the last elections owing to which the party won elections and became pro-governmental  $(N_{10} < N_{20}); t = T$  is the moment of the following, for example, parliamentary elections (as a rule T = 4 years or 1460 days if t will change on days);  $a_1(t)$ ,  $a_1(t)$  respectively factors of attraction of votes of oppositional and pro-governmental parties at the moment of time t, connected with the action program, financial and information possibilities (PR technology) of these parties;  $f(t, N_1(t))$  is the positive function of the arguments characterizing use of administrative resources, directed on voters of opposition party, for the purpose of their attraction on the party and power preservation that is the purpose of any authorities in power.

In model (1.1), (1.2) it is supposed that total number of voters  $(N_{10} + N_{20} = a)$  from elections to elections doesn't change (often, in many countries, their change is insignificant in comparison with a total number of voters). Thus, we consider that in a period between elections the number of the dead voters and the voters who for the first time have received a vote are equal (in many countries of 18 years) authorities in power.

This mathematical model doesn't consider falsification of elections in the election day though it is possible to consider and falsification cases, having initially set their percentage value.

Two cases are considered:

1.  $\alpha_1(t) = \alpha_1 = const > 0, \alpha_2(t) = \alpha_2 = const > 0, f(t, N_1(t)) = b > 0$  is constant nature of use of administrative resources.

2.  $\alpha_1(t) = \alpha_1 = const > 0, \alpha_2(t) = \alpha_2 = const > 0, f(t, N_1(t)) = \beta N_1(t), \beta > 0$ variable nature of use of administrative resources (in proportion to a number of voters of opposition party).

#### 2. Constant nature of use of administrative resources

In this case we have a system of the equations

$$\frac{dN_1(t)}{dt} = (\alpha_1 - \alpha_2)N_1(t)N_2(t) - b$$

$$\frac{dN_2(t)}{dt} = (\alpha_2 - \alpha_1)N_1(t)N_2(t) + b$$
(2.1)

depending on ratios between constants of model, the exact solution of a problem of Cauchy's (2.1), (1.2) look like: a)  $\alpha_1 < \alpha_2$ 

$$N_{1}(t) = \frac{a}{2} + \frac{p\left(1 + \frac{N_{10} - N_{20} - 2p}{N_{10} - N_{20} + 2p} \cdot \exp(2(\alpha_{2} - \alpha_{1})pt)\right)}{1 + \frac{N_{20} - N_{10} + 2p}{N_{10} - N_{20} + 2p} \cdot \exp(2(\alpha_{2} - \alpha_{1})pt)}$$

$$N_{2}(t) = \frac{a}{2} - \frac{p\left(1 + \frac{N_{10} - N_{20} - 2p}{N_{10} - N_{20} + 2p} \cdot \exp(2(\alpha_{2} - \alpha_{1})pt)\right)}{1 + \frac{N_{20} - N_{10} + 2p}{N_{10} - N_{20} + 2p} \cdot \exp(2(\alpha_{2} - \alpha_{1})pt)}$$

$$p = \sqrt{\frac{b}{\alpha_{2} - \alpha_{1}} + \frac{a^{2}}{4}} > \frac{a}{2}$$

$$(2.2)$$

$$\exp(2(\alpha_2 - \alpha_1)pt_1) = \frac{a+2p}{2p-a} \cdot \frac{N_{10} - N_{20} + 2p}{N_{20} - N_{10} + 2p} > 1$$
$$N_2(t_1) = a, \quad N_1(t_1) = 0$$
$$t_1 = \frac{1}{2(\alpha_2 - \alpha_1)p} \ln\left[\frac{a+2p}{2p-a} \cdot \frac{N_{10} - N_{20} + 2p}{N_{20} - N_{10} + 2p}\right].$$
(2.3)

If  $t_1 \leq T$ , then on the following elections the opposition party will have no voters supporting them (exponential aspiration to an one-party regime); if  $t_1 > T$ , then on the following elections opposition party will support only insignificant number of voters (close to a one-party regime)

$$N_1(T) = \frac{a}{2} + \frac{p\left(1 + \frac{N_{10} - N_{20} - 2p}{N_{10} - N_{20} + 2p} \cdot \exp(2(\alpha_2 - \alpha_1)pT)\right)}{1 + \frac{N_{20} - N_{10} + 2p}{N_{10} - N_{20} + 2p} \cdot \exp(2(\alpha_2 - \alpha_1)pT)} > 0$$
(2.4)

b)  $\alpha_1 = \alpha_2$ 

$$N_1(t) = -bt + N_{10} \quad N_2(t) = bt + N_{20}.$$
 (2.5)

It is clear, that in case of equality of factors of attraction of votes of competing parties, the number of voters of pro-governmental party grows, and oppositional falls, and, if

$$t_2 = \frac{N_{10}}{b} \le T,$$

then on the following elections the opposition party will have no voters supporting them (linear aspiration to an one-party regime). If

$$t_2 = N_{10}/b > T,$$

then on the following elections opposition party will support only an insignificant number of voters (close to a one - party regime).

$$N_1(T) = -bT + N_{10} > 0 (2.6)$$

c)  $\alpha_1 > \alpha_2$ 

c1) D = 0

$$D = \frac{a^2}{4} - \frac{b}{\alpha_1 - \alpha_2}$$
(2.7)

$$N_{1}(t) = \frac{a}{2} + \frac{N_{10} - N_{20}}{2 + (\alpha_{1} - \alpha_{2})(N_{10} - N_{20})t},$$

$$N_{2}(t) = \frac{a}{2} + \frac{N_{20} - N_{10}}{2 + (\alpha_{1} - \alpha_{2})(N_{10} - N_{20})t}.$$
(2.8)

The decision (2.8) is considered only at a period

$$t \in [0, t_3], t_3 = \frac{4N_{10}}{a(\alpha_1 - \alpha_2)(N_{20} - N_{10})} > 0, N_1(t_3) = 0, N_2(t_3) = a.$$
(2.9)

Therefore, if

 $t_3 \leq T$ ,

then on the following elections the opposition party will have no voters supporting them (hyperbolic aspiration to a one-party regime); if

 $t_3 > T$ ,

that at the following elections opposition party will support only an insignificant number of voters (close to a one - party regime)

$$N_1(T) = \frac{a}{2} + \frac{N_{10} - N_{20}}{2 + (\alpha_1 - \alpha_2)(N_{10} - N_{20})T} > 0$$
(2.10)

c2) D > 0

$$D = \frac{a^2}{4} - \frac{b}{\alpha_1 - \alpha_2} = q^2, q < \frac{a}{2}$$
(2.11)

$$N_{1}(t) = \frac{a}{2} + \frac{q\left(1 + \frac{N_{20} - N_{10} + 2q}{N_{20} - N_{10} - 2q} \cdot \exp(-2(\alpha_{1} - \alpha_{2})qt)\right)}{1 - \frac{N_{20} - N_{10} + 2q}{N_{20} - N_{10} - 2q} \cdot \exp(-2(\alpha_{1} - \alpha_{2})qt)},$$

$$N_{2}(t) = \frac{a}{2} - \frac{q\left(1 + \frac{N_{20} - N_{10} + 2q}{N_{20} - N_{10} - 2q} \cdot \exp(-2(\alpha_{1} - \alpha_{2})qt)\right)}{1 - \frac{N_{20} - N_{10} - 2q}{N_{20} - N_{10} - 2q} \cdot \exp(-2(\alpha_{1} - \alpha_{2})qt)}.$$
(2.12)

If the inequality is executed

$$N_{10} < N_{20} < N_{10} + 2q, (2.13)$$

then, at

$$t_4 = \frac{1}{2(\alpha_1 - \alpha_2)q} \ln \frac{N_{20} - N_{10} + 2q}{N_{10} + 2q - N_{20}}$$
(2.14)

the ratio takes place

$$N_1(t_4) = N_2(t_4),$$

then, at

$$t > t_4, \quad N_1(t) > N_2(t).$$
 (2.15)

Therefore, if  $t_4 < T$ , that the opposition party will win the following elections, a case

 $t_4 = T$ 

on the following elections both parties will collect identical quantities of votes, and at

 $t_4 > T,$ 

at the following elections at pro - governmental party all the same while will be voters more.

If equality takes place

$$N_{20} = N_{10} + 2q, (2.16)$$

that decision (2.12) will become

$$N_1(t) = N_{10}, \quad N_2(t) = N_{20},$$
 (2.17)

i.e. the number of voters of parties doesn't change over time and at the subsequent elections the pro - governmental party will keep the power.

At inequality performance

$$a > N_{20} > N_{10} + 2q$$

$$t_5 = \frac{1}{2(\alpha_1 - \alpha_2)q} \ln \frac{N_{20} - N_{10} + 2q}{N_{20} - N_{10} - 2q} \cdot \frac{N_{10} + N_{20} - 2q}{N_{20} + N_{10} + 2q} \qquad (2.18)$$

$$N_1(t_5) = 0, N_2(t_5) = a.$$

Therefore, if

 $t_5 \leq T$ ,

then at the following elections the opposition party will have no voters supporting them (exponential aspiration to a one - party regime); if

 $t_5 > T$ ,

that at the following elections opposition party will support only insignificant number of voters (close to a one - party regime)

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$$N_{1}(T) = \frac{a}{2} + \frac{q\left(1 + \frac{N_{20} - N_{10} + 2q}{N_{20} - N_{10} - 2q} \cdot \exp(-2(\alpha_{1} - \alpha_{2})qT)\right)}{1 - \frac{N_{20} - N_{10} + 2q}{N_{20} - N_{10} - 2q} \cdot \exp(-2(\alpha_{1} - \alpha_{2})qT)} > 0$$
(2.19)

c3) D < 0

$$D = \frac{a^2}{4} - \frac{b}{\alpha_1 - \alpha_2} = -r^2, \qquad (2.20)$$

$$N_{1}(t) = \frac{a}{2} - \frac{r\left(\frac{N_{20} - N_{10}}{2r} + \tan((\alpha_{1} - \alpha_{2})rt)\right)}{1 - \frac{N_{20} - N_{10}}{2r}\tan((\alpha_{1} - \alpha_{2})rt)}$$
(2.21)  
$$N_{2}(t) = \frac{a}{2} + \frac{r\left(\frac{N_{20} - N_{10}}{2r} + \tan((\alpha_{1} - \alpha_{2})rt)\right)}{1 - \frac{N_{20} - N_{10}}{2r}\tan((\alpha_{1} - \alpha_{2})rt)}$$

$$t_6 = \frac{1}{(\alpha_1 - \alpha_2)r} \operatorname{arctg} \frac{4rN_{10}}{N_{20}^2 - N_{10}^2 + 4r^2}$$
$$N_1(t_6) = 0, N_2(t_6) = a.$$

Therefore, if

 $t_6 \leq T$ ,

then on the following elections the opposition party will have no voters supporting them (transcendental aspiration to a one-party regime); if  $t_6 > T$ , that of the following elections opposition party will support only an insignificant number of voters (close to an one-party regime)

$$N_1(T) = \frac{a}{2} - \frac{r\left(\frac{N_{20} - N_{10}}{2r} + \tan((\alpha_1 - \alpha_2)rT)\right)}{1 - \frac{N_{20} - N_{10}}{2r}\tan((\alpha_1 - \alpha_2)rT)} > 0$$
(2.22)

## 3. Variable nature of use of administrative resources

In this case we have a system of the equations

$$\frac{dN_1(t)}{dt} = (\alpha_1 - \alpha_2)N_1(t)N_2(t) - \beta N_1(t), \qquad (3.1)$$

$$\frac{dN_{2}(t)}{dt} = (\alpha_{2} - \alpha_{1})N_{1}(t) N_{2}(t) + \beta N_{1}(t).$$

Depending on ratios between constants of model, the exact solution of a problem of Cauchy's (3.1), (1.2) look like:

a)  $\alpha_1 = \alpha_2$ 

$$N_1(t) = N_{10}e^{-\beta t}, \quad N_2(t) = a - N_{10}e^{-\beta t}.$$
 (3.2)

From (3.2) it is clear that in case of equality of factors of involvement of voters of competing parties, the number of voters of pro - governmental party grows, and oppositional falls and on the following elections it will support only an insignificant number of voters (exponential aspiration to a one - party regime)

$$N_1(T) = N_{10}e^{-\beta T}, (3.3)$$

b)  $\alpha_1 \neq \alpha_2, (\alpha_1 - \alpha_2)a = \beta, \alpha_1 > \alpha_2$ 

$$N_{1}(t) = \frac{N_{10}}{1 + (\alpha_{1} - \alpha_{2})N_{10}t},$$

$$N_{2}(t) = \frac{N_{20} + (\alpha_{1} - \alpha_{2})aN_{10}t}{1 + (\alpha_{1} - \alpha_{2})N_{10}t}$$
(3.4)

From (3.4) it follows that if a number of voters of pro - governmental party grows, and oppositional falls and at the following elections it will support only an insignificant number of voters (hyperbolic aspiration to a one - party regime).

c) $\alpha_1 \neq \alpha_2, (\alpha_1 - \alpha_2)a \neq \beta$ 

$$N_{1}(t) = \frac{((\alpha_{2} - \alpha_{1})a + \beta)N_{10}e^{-((\alpha_{2} - \alpha_{1})a + \beta)t}}{(\alpha_{2} - \alpha_{1})N_{20} + \beta + (\alpha_{2} - \alpha_{1})N_{10}e^{-((\alpha_{2} - \alpha_{1})a + \beta)t}}$$
(3.5)  
$$N_{2}(t) = \frac{(\alpha_{2} - \alpha_{1})aN_{20} + a\beta - \beta N_{10}e^{-((\alpha_{2} - \alpha_{1})a + \beta)t}}{(\alpha_{2} - \alpha_{1})N_{20} + \beta + (\alpha_{2} - \alpha_{1})N_{10}e^{-((\alpha_{2} - \alpha_{1})a + \beta)t}}$$

c1)  $\alpha_1 < \alpha_2$  From (3.5) it follows that in this case, the number of voters of progovernmental party grows, and oppositional falls and at the following elections it will support only an insignificant number of voters (exponential aspiration to a one - party regime)

$$N_1(T) = \frac{((\alpha_2 - \alpha_1)a + \beta)N_{10}e^{-((\alpha_2 - \alpha_1)a + \beta)T}}{(\alpha_2 - \alpha_1)N_{20} + \beta + (\alpha_2 - \alpha_1)N_{10}e^{-((\alpha_2 - \alpha_1)a + \beta)T}}$$
(3.6)

c2)  $\alpha_1 > \alpha_2, 0 < (\alpha_1 - \alpha_2)a < \beta$  From (3.5) it follows that in this case, the number of voters of pro - governmental party grows, and oppositional falls and at the following elections it will support only an insignificant number of voters (exponential aspiration to a one - party regime)

$$N_1(T) = \frac{(\beta - (\alpha_1 - \alpha_2)a)N_{10}e^{-(\beta - (\alpha_1 - \alpha_2)a)T}}{\beta - (\alpha_2 - \alpha_1)N_{20} - (\alpha_2 - \alpha_1)N_{10}e^{-(\beta - (\alpha_1 - \alpha_2)a)T}}$$
(3.7)

c3)  $\alpha_1 > \alpha_2, (\alpha_1 - \alpha_2)a > \beta.$ 

Let's introduce the notation

$$g(t) \equiv (\alpha_1 - \alpha_2)N_{20} - \beta + (\alpha_1 - \alpha_2)N_{10}e^{((\alpha_1 - \alpha_2)a - \beta)t}.$$
(3.8)

It is easy to show that we have

$$g'(t) > 0, \quad g(0) > 0.$$

Therefore owing to a g(t) function continuity

$$g(t) > 0, \text{ for } t > 0$$

If the inequality takes place

$$(\alpha_1 - \alpha_2)a > 2\beta, \tag{3.9}$$

then inequalities are fair

$$(\alpha_1 - \alpha_2)N_{20} > 2\beta, \quad \frac{((\alpha_1 - \alpha_2)N_{20} - \beta)a}{N_{10}((\alpha_1 - \alpha_2)a - 2\beta)} > 1,$$
 (3.10)

and also an inequality for required functions

$$N_1(t) \ge N_2(t), \quad t \ge t_7,$$
 (3.11)

$$t_7 = \frac{1}{(\alpha_1 - \alpha_2)a - \beta} \ln \frac{((\alpha_1 - \alpha_2)N_{20} - \beta)a}{N_{10}((\alpha_1 - \alpha_2)a - 2\beta)}.$$

Therefore, if

$$t_7 < T$$
,

then the opposition party will win the following elections, a case

$$t_7 = T$$

at the following elections both parties will collect identical quantities of votes, and at

$$t_7 > T$$
,

at the following elections at pro - governmental party all the same while will be voters more.

In the case

$$\beta < (\alpha_1 - \alpha_2)a \le 2\beta,$$
  
$$N_1(t) < N_2(t), \quad t \ge 0$$

and the opposition party will lose the following elections.

The mathematical model except theoretical interest has also important practical value, as both parties (the state structures in together with pro - governmental party; opposition party) can use results according to the purposes. It allows the parties, according to the chosen strategy, to select parameters of action and to achieve desirable results for them.

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