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THE SOLUTION OF THE STRESS PROBLEM OF THE THEORY OF THERMOELASTICITY WITH MICROTEMPERATURES FOR A CIRCULAR RING

Tsagareli I., Svanadze M.

Abstract. The solution of statics of the stress boundary value problem of the theory of thermoelasticity with microtemperatures for the circular ring is presented. The representation of regular solutions for the system of equations of the linear theory of thermoelasticity with microtemperatures by harmonic, biharmonic and metaharmonic functions is obtained. The solution is obtained by means of absolutely and uniformly convergent series. The question on the uniqueness of the solution of the problem is studied.

Keywords and phrases: Thermoelasticity, microtemperature, sress problem, uniqueness theorem, explicit solutions.

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1. Basic equations

The basic system of equations of the theory of thermoelasticity with microtemperatures can be written in the form [1,2]:

$$\begin{aligned} &\mu\Delta u(x) + (\lambda+\mu) graddivu(x) = \beta gradu_3(x),\\ &k\Delta u_3(x) + k_1 divw(x) = 0, \\ &k_6\Delta w(x) + (k_4+k_5) graddivw(x) - k_3 gradu_3(x) - k_2 w(x) = 0, \end{aligned} \tag{1}$$

where $\lambda, \mu, \beta, k, k_1, k_2, k_3, k_4, k_5, k_6$ are constitutive coefficients [1]; $u(x)$ is the displacement vector of the point $x = (x_1, x_2)$; $u = (u_1, u_2)$; $w = (w_1, w_2)$ is the microtemperatures vector; u_3 is temperature measured from the constant absolute temperature T_0; Δ is the Laplace operator.

Problem. Find a regular vector $U = (u_1, u_2, u_3, w_1, w_2)$, ($U \in C^1(\overline{D}) \cap C^2(D), \overline{D} = D \cup S_0 \cup S_1$) satisfying in the ring D a system of equations (1) and on the circumferences S_0 and S_1 the boundary conditions:

$$\begin{aligned} &[T'(\partial_z, n)u(z) - \beta u_3(z)n(z)]^i = f^i(z), \quad \left[k\frac{\partial u_3(z)}{\partial n(z)} + k_1 w(z)n(z)\right]^i = f_3^i(z), \\ &[T''(\partial_z, n)w(z)]^i = p^i(z), \quad i = 0, 1, \end{aligned} \tag{2}$$

where $f = (f_1, f_2)$, $p = (p_1, p_2)$, f_1, f_2, f_3 are the given functions on S_0 and S_1; $T'u$ is the stress vector in the classical theory of elasticity; $T''w$ is stress vector for microtemperatures [1,2]:

$$
\begin{aligned}
&T'(\partial_x,n)u(x)=\mu\frac{\partial u(x)}{\partial n}+\lambda n(x)divu(x)+\mu\sum_{i=1}^{2}n_i(x)gradu_i(x),\\
&T''(\partial_x,n)w(x)=(k_5+k_6)\frac{\partial w(x)}{\partial n}+k_4n(x)divw(x)+k_5\sum_{i=1}^{2}n_i(x)gradw_i(x).
\end{aligned}
\tag{3}
$$

The above-formulated problem of thermoelasticity with microtemperatures can be considered as a union of two problems A and B, where:

Problem A. find in a ring D the solution $u(x)$ of equation $(1)_1$, if on the circumferences S_0 and S_1 there are given the values of the vector $T'(\partial_z,n)u(z)-\beta u_3(z)n(z)$;

Problem B. find in the ring D the solutions $u_3(x)$ and $w(x)$ of the system of equations $(1)_2$ and $(1)_3$,if on the circumferences S_0 and S_1 there are given the values of the function $k\dfrac{\partial u_3(z)}{\partial n(z)}+k_1w(z)n(z)$ and of the vector $T''(\partial_z,n)w(z)$.

Let (u',u_3',w') and (u'',u_3'',w'') be two different solutions of any of the problems. Then the differences $u=u'-u''$, $\quad u_3=u_3'-u_3''$ $\quad$ and $w=w'-w''$ of these solutions, obviously, satisfies the homogeneous system $(1)_0$ and zero boundary conditions $(2)_0$. For a regular solutions of equation $(1)_1$ and equations $(1)_2$ and $(1)_3$ the Green's formulas [2,3]:

$$
\int_D[E_1(u(x),u(x))-\beta u_3(x)divu(x)]dx=-\int_S u^0(y)[T'(\partial_y,n)u(y)-\beta u_3(y)n(y)]^0d_yS_0
$$

$$
+\int_S u^1(y)[T'(\partial_y,n)u(y)-\beta u_3(y)n(y)]^1d_yS_1,
$$

$$
\begin{aligned}
&\int_D[T_0E_2(w(x),w(x))+k\mid gradu_3\mid^2+(k_1+k_3T_0)wgradu_3+k_2T_0\mid w(x)\mid^2]dx\\
&=-\textstyle\int_S u_3^0(y)[k\partial_n u_3(y)+k_1w(y)n(y)]^0+T_0w^0(y)[T''(\partial_y,n)w(y)]^0d_yS_0\\
&+\textstyle\int_S u_3^1(y)[k\partial_n u_3(y)+k_1w(y)n(y)]^1+T_0w^1(y)[T''(\partial_y,n)w(y)]^1d_yS_1
\end{aligned}
\tag{4}
$$

is valid, where

$$
E_1(u,u)=(\lambda+\mu)(\partial_1u_1+\partial_2u_2)^2+\mu(\partial_1u_1-\partial_2u_2)^2+\mu(\partial_2u_1+\partial_1u_2)^2;
$$

$$
\begin{aligned}
E_2(w,w)=&\frac{1}{2}(2k_4+k_5+k_6)(\partial_1w_1+\partial_2w_2)^2+(k_6+k_5)(\partial_1w_1-\partial_2w_2)^2\\
&+(k_6+k_5)(\partial_2w_1+\partial_1w_2)^2+(k_6-k_5)(\partial_1w2-\partial_2w_1)^2,
\end{aligned}
$$

under the conditions that: $\lambda+\mu$, $\quad\mu>0$ and, accordingly, $2k_4+k_5+k_6>0, k_6\pm k_5>0$ are nonnegative quadratic forms.

Taking into account formula $(4)_2$ under the homogeneous boundary conditions for the problem B, we obtain $E_2(w,w)=0$, $\quad gradu_3=0$, $\quad w=0$. The solution of the above equations has the form: $u_3(x)=const, w=0$.

The following uniqueness theorem is valid.

Theorem. *The difference of two arbitrary solutions of the BVP (1), (2) is the vector* $U = (u_1, u_2, u_3, w_1, w_2)$*, where* $u_1(x) = -c_1x_2 + clx_1 + q_1, u_2(x) = -c_1x_1 + clx_1 + q_2, u_3 = c, w_1 = w_2 = 0; c, c_1, q_1, q_2$ *are arbitrary constants,* $l = \frac{\beta}{2(\lambda+\mu)}$.

2. Solution of the problem B

Taking into account formulas: $\dfrac{\partial}{\partial x_2} = n_2\dfrac{\partial}{\partial r} + \dfrac{n_1}{r}\dfrac{\partial}{\partial \psi}, \quad \dfrac{\partial}{\partial x_1} = n_1\dfrac{\partial}{\partial r} - \dfrac{n_2}{r}\dfrac{\partial}{\partial \psi}$, we rewrite the representation solutions of the system $[(1)_2, (1)_3]$ and the boundary conditions of the problem B in the tangent and normal components [3]:

$$\begin{aligned} &u_3(x) = \varphi_1(x) + \varphi_2(x),\\ &w_n(x) = a_1\frac{\partial}{\partial r}\varphi_1(x) + a_2\frac{\partial}{\partial r}\varphi_2(x) - a_3\frac{1}{r}\frac{\partial}{\partial \psi}\varphi_3(x),\\ &w_s(x) = a_1\frac{1}{r}\frac{\partial}{\partial \psi}\varphi_1(x) + a_2\frac{1}{r}\frac{\partial}{\partial \psi}\varphi_2(x) + a_3\frac{\partial}{\partial r}\varphi_3(x), \end{aligned} \tag{5}$$

$$\begin{aligned} &k\left[\frac{\partial u_3}{\partial r}\right]^i + k_1\left[w_n\right]^i = f_3^i(z), \quad k_7\left[\frac{\partial w_n}{\partial r}\right]^i + \frac{k_4}{R_i}\left[\frac{\partial w_s}{\partial \psi}\right]^i = p_n^i(z),\\ &k_6\left[\frac{\partial w_s}{\partial r}\right]^i + \frac{k_5}{R_i}\left[\frac{\partial w_n}{\partial \psi}\right]^i = p_s^i(z), \end{aligned} \tag{6}$$

where $w_n = (w \cdot n), w_s = (w \cdot s), p_n = (p \cdot n), p_s = (p \cdot s), n = (n_1, n_2), s = (-n_2, n_1)$, $\dfrac{\partial}{\partial n} = \dfrac{\partial}{\partial r}, \quad i = 0, 1; \quad \triangle\varphi_1 = 0, (\triangle + s_1^2)\varphi_2 = 0, (\triangle + s_2^2)\varphi_3 = 0, s_1^2 = -\dfrac{kk_2 - k_1k_3}{kk_7}$, $s_2^2 = -\dfrac{k_2}{k_6}, a_1 = -\dfrac{k_3}{k_2}, a_2 = -\dfrac{k}{k_1}, a_3 = \dfrac{k_6}{k_7}; \quad k_7 = k_4 + k_5 + k_6; \quad k, k_2, k_6, k_7 > 0$; $w_n = (w \cdot n), \quad w_s = (w \cdot s), \quad p_n = (p \cdot n), \quad p_s = (p \cdot s), \quad n = (n_1, n_2), \quad s = (-n_2, n_1); \quad x = (r, \psi), \quad r^2 = x_1^2 + x_2^2$. R_0 is radius of the boundary S_0; R_1 is radius of the boundary S_1.

The harmonic function φ_1 and metaharmonic functions φ_2 and φ_3 are represented in the form of series in the ring ([4], p.417; [5]):

$$\begin{aligned} &\varphi_1(x) = X_{10}\ln r + Y_{10} + \sum_{m=1}^{\infty}[r^m(X_{1m} \cdot \nu_m(\psi)) + r^{-m}(X_{1m} \cdot \nu_m(\psi))],\\ &\varphi_2(x) = \sum_{m=0}^{\infty}[I_m(s_2r)(X_{2m} \cdot \nu_m(\psi)) + K_m(s_2r)(Y_{2m} \cdot \nu_m(\psi))],\\ &\varphi_3(x) = \sum_{m=0}^{\infty}[I_m(s_3r)(X_{3m} \cdot \ s_m(\psi)) + K_m(s_3r)(Y_{3m} \cdot \ s_m(\psi))], \end{aligned} \tag{7}$$

where $I_m(s_jr)$ and $K_m(s_jr)$ are the Bessel's and modified Hankel's functions of an imaginary argument, respectively; X_{km} and Y_{km} are the unknown two-component constants vectors, $\nu_m(\psi) = (\cos m\psi, \sin m\psi), s_m(\psi) = (-\sin m\psi, \cos m\psi), j = 2, 3; k = 1, 2$.

We substitute (7) into (5) and then the obtained expressions substitute into (6). Passing to the limit, as $r \to R_0$ and $r \to R_1$ for the unknowns X_{mk} and Y_{mk} we obtain a system of algebraic equations:

$$-a_1\frac{1}{R_i^2}X_{10}+a_2s_2^2[I_0''(s_2R_i)X_{20}+K_0''(s_2R_i)Y_{20}]=\frac{A_{10}^i}{2k_7},$$
$$I_0''(s_3R_i)X_{30}+K_0''(s_3R_i)Y_{30}=\frac{A_{20}^i}{2k_6a_3s_3},$$
$$\frac{1}{R_i}(1+k_1a_1)X_{10}+s_2(1+a_2)[I_0'(s_2R_i)X_{20}+K_0'(s_2R_i)Y_{20}]=\frac{A_{30}^i}{2},$$

$$a_1mR_i^{m-2}[k_7(m-1)-k_4m]X_{1m}+a_2\left[k_7s_2^2I_m''(s_2R_i)-k_4\frac{m^2}{R_i^2}I_m(s_2R_i)\right]X_{2m}$$
$$+k_7a_3\frac{m}{R_i}\left[\frac{1}{R_i}I_m(s_3R_i)+s_3I_m'(s_3R_i)\right]X_{3m}+a_1mR_i^{-(m+2)}\left[k_7(m+1)-k_4m\right]Y_{1m}$$
$$+a_2\left[k_7s_2K_m''(s_2R_i)-k_4\frac{m^2}{R_i^2}K_m(s_2R_i)\right]Y_{2m}$$
$$+k_7a_3\frac{m}{R_i}\left[\frac{1}{R_i}K_m(s_3R_i)+s_3K_m'(s_3R_i)\right]Y_{3m}=A_{1m}^i,$$

$$a_1mR_i^{m-2}[k_5m+k_6(m-1)]X_{1m}+a_2\frac{m}{R_i}\left[-k_6\frac{1}{R_i}I_m(s_2R_i)+s_2(k_5+k_6)I_m'(s_2R_i)\right]X_{2m}$$
$$+a_3[k_6s_3^2I_m''(s_3R_i)-k_5\frac{m^2}{R_i^2}I_m(s_3R_i)]X_{3m}-a_1mR_i^{-(m+2)}[k_6(m+1)+k_5m]Y_{1m}$$
$$+a_2\frac{m}{R_i}\left[-k_6\frac{1}{R_i}K_m(s_2R_i)+(k_5+k_6)s_2K_m'(s_2R_i)\right]Y_{2m}$$
$$+a_3\left[-k_5\frac{m^2}{R_i^2}K_m(s_3R_i)+k_6s_3^2K_m''(s_3R_i)\right]Y_{3m}=A_{2m}^i,$$

$$k_1mR_i^{m-1}X_{1m}+s_2I_m'(s_2R_i)(k+k_1a_2)X_{2m}-k_1a_3\frac{m}{R_i}I_m(s_3R_i)X_{3m}$$
$$-k_1mR_i^{-(m+1)}Y_{1m}+s_2(k+k_1a_2)K_m'(s_2R_i)Y_{2m}-k_1a_3\frac{m}{R_i}K_m(s_3R_i)Y_{3m}=A_{3m}^i,$$

where A_{1m}^i, A_{2m}^i and A_{3m}^i are the Fourier coefficients of the functions $p_n(z)$, $p_s(z)$ and $f_3(z)$, respectively; i=0,1; m=1,2,....

3. Solution of the problem A

The solution of the first equation of the system (1) with the boundary condition (2) is represented by the sum

$$u(x)=v_0(x)+v(x), \tag{8}$$

where v_0 is a particular solution of equation $(1)_1$:

$$v_0(x)=\frac{\beta}{\lambda+2\mu}grad[-\frac{1}{s_1^2}\varphi_2(x)+\varphi_0(x)];$$

φ_0 is a biharmonic function: $\triangle\varphi_0 = \varphi_1$; $v(x) = (v_1(x), v_2(x))$ is the solution of the homogeneous equation $\mu\triangle v(x) + (\lambda+\mu) graddivv(x) = 0$ which can be found by means of the formulae [6]

$$v_1(x) = \frac{\partial}{\partial x_1}[\Phi_1(x) + \Phi_2(x)] - \frac{\partial}{\partial x_2}\Phi_3(x), \quad v_2(x) = \frac{\partial}{\partial x_2}[\Phi_1(x) + \Phi_2(x)] + \frac{\partial}{\partial x_1}\Phi_3(x),$$

where $\Delta\Phi_1(x) = 0, \quad \Delta\Delta\Phi_2(x) = 0, \quad \Delta\Delta\Phi_3(x) = 0;$

$$\begin{aligned}
\Phi_1(x) &= \sum_{m=1}^{\infty}\left[\left(\frac{r}{R_1}\right)^m (Z_{1m}\cdot\nu_m(\psi)) + \left(\frac{R_0}{r}\right)^m (Z_{2m}\cdot\nu_m(\psi))\right] + Z_{10}\ln r\\
\Phi_2(x) &= \sum_{m=0}^{\infty}\left(\frac{r}{R_1}\right)^{m+2}(Z_{3m}\cdot\nu_m(\psi))\\
&+\sum_{m=2}^{\infty}\left(\frac{R_0}{r}\right)^{m-2}(Z_{4m}\cdot\nu_m(\psi)) + r\ln r(Z_{41}\cdot\nu_1(\psi)) + \frac{1}{2}\left(\frac{r}{R_1}\right)^2 Z_{20}\\
\Phi_3(x) &= -\frac{(\lambda+2\mu)}{\mu}\sum_{m=1}^{\infty}\left(\frac{r}{R_1}\right)^{m+2}(Z_{3m}\cdot s_m(\psi))\\
&+\frac{\lambda+2\mu}{\mu}\sum_{m=2}^{\infty}\left(\frac{R_0}{r}\right)^{m-2}(Z_{4m}\cdot s_m(\psi))\\
&+\frac{(\lambda+2\mu)}{\mu} r\ln r(Z_{11}\cdot s_1(\psi)) + Z_{40}\ln r + \frac{1}{2}\left(\frac{r}{R_1}\right)^2 Z_{30},
\end{aligned} \tag{9}$$

where Z_{km} are the unknown two-component vectors, $k = 1,2,3,4$. Taking into account (8) and relying on the condition $(2)_I$, we can write

$$[T'(\partial_z, n)v(z)]^i = \Psi^i(z), \quad z\in S_i, \quad i = 0,2,$$

where $\Psi^i(z) = f^i(z) + \beta u_3^i(z)n(z) - [T'(\partial_z, n)v_0(z)]^i$ is the known vector, $\Psi^i = (\Psi_1^i, \Psi_2^i)$. We rewrite this conditions in the tangent and normal components:

$$[T'(\partial_z, n)v(z)]_n^i = \Psi_n^i(z), \quad [T'(\partial_z, n)v(z)]_s^i = \Psi_s^i(z), \tag{10}$$

where

$$[T'(\partial_z, n)v(z)]_n^i = (\lambda+2\mu)\frac{\partial v_n^i(z)}{\partial r} + \lambda\frac{1}{R_i}\frac{\partial v_s^i(z)}{\partial\psi},$$

$$[T'(\partial_z, n)v(z)]_s^i = \mu\frac{\partial v_s^i(z)}{\partial r} + \mu\frac{1}{R_i}\frac{\partial v_n^i(z)}{\partial\psi},$$

$$v_n^i(z) = \frac{\partial}{\partial r}(\Phi_1^i(z) + \Phi_2^i(z)) - \frac{1}{r}\frac{\partial}{\partial\psi}\Phi_3^i(z),$$

$$v_s^i(z) = \frac{1}{r}\frac{\partial}{\partial\psi}(\Phi_1^i(z) + \Phi_2^i(z)) + \frac{\partial}{\partial r}(\Phi_3^i(z)).$$

We substitute (9) into (10). Passing to the limit, as $r \to R_0$ and $r \to R_1$ for the unknowns Z_{mk} we obtain a system of algebraic equations:

$$A(m)t^{m-2}Z_{1m} + B(m)Z_{2m} + C(m)t^m Z_{3m} + E_1(m)Z_{4m} = \eta_m^0,$$
$$A(m)Z_{1m} + B(m)t^{m+2}Z_{2m} + C(m)Z_{3m} + E_2(m)Z_{4m} = \eta_m^1,$$
$$A(m)t^{m-2}Z_{1m} + B(m)Z_{2m} + D(m)t^m Z_{3m} + E_3(m)Z_{4m} = \zeta_m^0,$$
$$A(m)Z_{1m} + B(m)t^{m+2}Z_{2m} + D(m)Z_{3m} + E_4(m)Z_{4m} = \zeta_m^1,$$

where

$$t = \frac{R_0}{R_1}, \quad e_1(m) = 2(\lambda+\mu)(m+1), \quad e_2(m) = 2(\lambda+\mu)(m-1),$$
$$A(m) = \frac{2\mu(m-1)m}{R_1^2}, \quad B(m) = \frac{2\mu(m+1)m}{R_0^2}, \quad C(m) = -\frac{e_{1m}(m-2)}{R_1^2},$$
$$D(m) = -\tfrac{e_1(m)m}{R_1^2}, \quad E_1(1) = \frac{2(2\lambda+3\mu)}{R_0}, \quad E_1(m) = -\frac{e_2(m)(m+2)}{R_0^2},$$
$$E_2(1) = \frac{2(2\lambda+3\mu)}{R_1}, \quad E_2(m) = -\frac{e_2(m)(m+2)}{\mu R_0}t^m, \quad E_3(1) = \frac{2\mu}{R_0}, \quad E_3(m) = \frac{e_2(m)m}{R_0^2},$$
$$E_4(1) = \frac{2\mu}{R_1}\ln R_1, \quad E_4(m) = \frac{e_2(m)t^m}{\mu R_0}, \quad m = 2, 3, ...$$

If the principal vector and principal moment of external stresses is equal to zero, then we obtain

$$R_1^2 \int_0^{2\pi} \Psi_s^1(\theta)d\theta - R_0^2 \int_0^{2\pi} \Psi_s^0(\theta)d\theta = 0.$$

From here when $m = 0$, we get: $R_1^2\zeta_0^1 = R_0^2\zeta_0^0$. When $m = 0$ for the unknowns Z_{10}, Z_{20} and Z_{40} we obtain the system

$$-\frac{2\mu}{R_i^2}Z_{10} + \frac{2(\lambda+\mu)}{R_1^2}Z_{20} = \frac{\zeta_0^i}{2}, \quad -\frac{2\mu}{R_0^2}Z_{40} = \frac{\zeta_0^0}{2},$$

Z_{30} is an arbitrary constant, $i = 0, 1$.

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Authors' addresses:

I. Tsagareli
I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: i.tsagareli@yahoo.com

M. Svanadze
Institute of Mathematics, University of Gottingen
3-5, Bunsenstrasse, Gottingen, D-37078
Germany
E-mail: maia.svanadze@gmail.com