Seminar of I. Vekua Institute of Applied Mathematics
REPORTS, Vol. 38, 2012

ON A CONNECTION BETWEEN CONTROLLABILITY OF THE INITIAL AND PERTURBED TWO-STAGE SYSTEMS

Tadumadze T.

Abstract. Sufficient conditions are established, guaranteeing controllability of the initial two-stage system of ordinary differential equations if a sequence of the perturbed two-stage systems is controllable, when the perturbations of right-hand side of system are small in the integral sense.

Keywords and phrases: Two-stage system, controllability, perturbations, sufficient conditions.

AMS subject classification (2010): 93B05, 93B12, 93C15, 93C73.

1. Formulation of main results

Let $t_{01} < t_{02} < \theta_1 < \theta_2 < t_{11} < t_{12}$ be given numbers and R^n_x be the n-dimensional vector space of points

$$x = (x^1, ..., x^n)^T, |x|^2 = \sum_{i=1}^{n} (x^i)^2,$$

where T means transpose; suppose that $O \subset R^n_x$ and $Y \subset R^m_y$ are open sets, $U \subset R^p_u$ and $V \subset R^q_v$ are compact sets. Further, let $E_f = E_f(I_1 \times O, R^n_x)$, be the space of functions $f(t,x) \in R^n_x$ defined on $I_1 \times O$ and satisfying the following conditions:

1.1. for any $x \in O$ the function $f(t,x)$ is measurable on $I_1 = [t_{01}, \theta_2]$;

1.2. for any function $f \in E_f$ and any compact set $K \subset O$ there exist functions $m_{f,K}(\cdot), L_{f,K}(\cdot) \in L_1(I_1, R_+), R_+ = [0, \infty)$ such that for almost all $t \in I_1$,

$$|f(t,x)| \leq m_{f,K}(t), \forall x \in K$$

and

$$|f(t,x_1) - f(t,x_2)| \leq L_{f,K}(t)|x_1 - x_2|, \forall (x_1, x_2) \in K^2.$$

Let $E^u_f = E^u_f(I_1 \times O, R^n_x)$ be the space of functions $f(t,x,u) \in R^n_x$ defined on $I_1 \times O \times U$ and satisfying the following conditions:

1.3. for any $(x,u) \in O \times U$ the function $f(t,x,u)$ is measurable on I_1;

1.4. for any function $f \in E^u_f$ and any compact set $K \subset O$ there exist functions $m_{f,K}(\cdot), L_{f,K}(\cdot) \in L_1(I_1, R_+)$ such that for almost all $t \in I_1$,

$$|f(t,x,u)| \leq m_{f,K}(t), \forall (x,u) \in K \times U$$

and

$$|f(t,x_1,u) - f(t,x_2,u)| \leq L_{f,K}(t)|x_1 - x_2|, \forall (x_1, x_2, u) \in K^2 \times U.$$

Analogously are defined the following spaces $E_g = E_g(I_2 \times Y, R^m_y)$ and $E^v_g = E^v_g(I_2 \times Y \times V, R^m_y)$, where $I_2 = [\theta_1, t_{12}]$.

Let $f_0 \in E_f^u$ and $g_0 \in E_g^v$ be given functions and $x_0 \in O$ and $y_1 \in Y$ be given points. By Ω and Δ we denote sets of measurable functions $u : I_1 \to U$ and $v : I_2 \to V$, respectively.

To each element

$$w = (t_0, \theta, t_1, u(\cdot), v(\cdot)) \in W = [t_{01}, t_{02}] \times [\theta_1, \theta_2] \times [t_{11}, t_{12}] \times \Omega \times \Delta$$

we assign the two-stage system of differential equations

$$\begin{cases} \dot{x} = f_0(t, x, u(t)), t \in [t_0, \theta], \\ \dot{y} = g_0(t, y, v(t)), t \in [\theta, t_1] \end{cases} \tag{1.1}$$

with the initial condition

$$x(t_0) = x_0 \tag{1.2}$$

and the intermediate condition at the switching moment θ

$$y(\theta) = Q(\theta, x(\theta)). \tag{1.3}$$

Here the function $Q(t, x) \in R_y^m$ is continuous on $[\theta_1, \theta_2] \times O$ and continuously differentiable with respect to $x \in 0$.

Definition 1.1. Let $w = (t_0, \theta, t_1, u(\cdot), v(\cdot)) \in W$. The pair of functions $\Phi(w) = \{x(t) = x(t; w) \in O, t \in [t_0, \theta]; y(t) = y(t; w) \in Y, t \in [\theta, t_1]\}$ is called solution corresponding to the element w, if the conditions (1.2) and (1.3) are fulfilled. Moreover, the function $x(t)$ is absolutely continuous and satisfies the first equation of (1.1) almost everywhere (a.e.) on $[t_0, \theta]$; the function $y(t)$ is absolutely continuous and satisfies the second equation of (1.1) a.e. on $[\theta, t_1]$.

Definition 1.2. The element $w \in W$ is admissible if for corresponding solution $\Phi(w)$ the condition

$$y(t_1) = y_1 \tag{1.4}$$

holds.

The set of admissible elements is denoted by W_0.

Definition 1.3. The system (1.1) is called controllable with the conditions (1.2)-(1.4), if $W_0 \neq \emptyset$.

To formulate the main results we introduce the following notation: let $C > 0, N > 0$ and $K \subset O, M \subset Y$ be given numbers and compact sets,

$$F_{K,C} = \Big\{ f \in E_f : \int_{I_1} [m_{f,K}(t) + L_{f,K}(t)] dt \leq C \Big\},$$

$$V_{K,\delta} = \Big\{ f \in F_{K,C} : \Big| \int_{s_1}^{s_2} f(t, x) dt \Big| \leq \delta, \forall s_1, s_2 \in I_1, x \in K \Big\}, \delta > 0;$$

$$G_{M,N} = \Big\{ g \in E_g : \int_{I_2} [m_{g,M}(t) + L_{g,M}(t)] dt \leq N \Big\},$$

$$H_{M,\delta}=\Big\{g\in G_{M,N}:\Big|\int_{s_1}^{s_2}g(t,y)dt\Big|\le\delta,\forall s_1,s_2\in I_2,y\in M\Big\};$$

$$F^u_{K,C}=\Big\{f\in E^u_f:\int_{I_1}[m_{f,K}(t)+L_{f,K}(t)]dt\le C\Big\},$$

$$V^u_{K,\delta}=\Big\{f\in F^u_{K,C}:\int_{I_1}\sup_{(x,u)\in K\times U}|f(t,x,u)|dt\le\delta\Big\},$$

$$G^v_{M,N}=\Big\{g\in E^v_g:\int_{I_2}[m_{g,M}(t)+L_{g,M}(t)]dt\le N\Big\},$$

$$H^v_{M,\delta}=\Big\{g\in G^v_{M,N}:\int_{I_2}\sup_{(y,v)\in M\times V}|g(t,y,v)|dt\le\delta\Big\};$$

$$B_{y_1,\varepsilon}=\Big\{y\in Y:\parallel y_1-y\parallel\le\varepsilon\Big\},\varepsilon>0.$$

Theorem 1.1. *Let the system (1.1) be controllable i.e. there exists $w_0=(t_{00},\theta_0,t_{10},u_0(\cdot),v_0(\cdot))\in W_0$. Then for arbitrary $\varepsilon>0$ there exists a number $\delta=\delta(\varepsilon)>0$ such that for any $f\in V_{K_{01},\delta}$ and $g\in H_{M_{01},\delta}$ the perturbed two-stage system*

$$\begin{cases}\dot{x}(t)=f_0(t,x,u(t))+f(t,x),t\in[t_0,\theta],\\ \dot{y}(t)=g_0(t,y,v(t))+g(t,y),t\in[\theta,t_1]\end{cases}\tag{1.5}$$

with the conditions

$$x(t_0)=x_0,y(\theta)=Q(\theta,x(\theta)),y(t_1)\in B_{y_1,\varepsilon}\tag{1.6}$$

is controllable. Here $K_{01}\subset O$ and $M_{01}\subset Y$ are compact sets, containing some neighborhoods of $K_0=\{x(t;w_0):t\in[t_{00},\theta_0]\}$ and $M_0=\{y(t;w_0):t\in[\theta_0,t_{10}]\}$, respectively.

Theorem 1.2. *Let the system (1.1) be controllable. Then for arbitrary $\varepsilon>0$ there exists a number $\delta=\delta(\varepsilon)>0$ such that for any $f\in V^u_{K_{01},\delta}$ and $g\in H^v_{M_{01},\delta}$ the perturbed two-stage system*

$$\begin{cases}\dot{x}(t)=f_0(t,x,u(t))+f(t,x,u(t)),t\in[t_0,\theta],\\ \dot{y}(t)=g_0(t,y,v(t))+g(t,y,v(t)),t\in[\theta,t_1]\end{cases}$$

with the conditions (1.6) is controllable.

Definition 1.4. The pair of functions $\hat{\Phi}(w)=\{\hat{x}(t)=\hat{x}(t;w)\in O,t\in I_1;\hat{y}(t)=\hat{y}(t;w)\in Y,t\in I_2\}$ is called a continuation of the solution $\Phi(w)$, if $\hat{x}(t)$ on the interval I_1 is a continuation of the solution $x(t),t\in[t_0,\theta]$ and $\hat{y}(t)$ on the interval I_2 is a continuation of the solution $y(t),t\in[\theta,t_1]$ (see Definition 1.1).

Theorem 1.3. *Let the following conditions hold:*

1.5. *for any $w\in W$ there exists the continuation solution $\hat{\Phi}(w)$; moreover, there exist compact sets $K_1\subset O$ and $M_1\subset Y$ such that, for any $w\in W$*

$$\hat{x}(t;w)\in K_1,t\in I_1\ \text{ and }\ \hat{y}(t;w)\in M_1,t\in I_2;$$

1.6. *the sets*

$$f_0(t,x,U)=\{f_0(t,x,u):u\in U\} \text{ for any fixed } (t,x)\in I_1\times O$$

and

$$g_0(s,y,V)=\{g_0(t,x,v):v\in V\} \text{ for any fixed } (s,y)\in I_2\times Y$$

are convex;

1.7. *there exist sequences* $\{\varepsilon_i\}\to 0, \{\delta_i\}\to 0, \{f_i\in V_{K_{11},\delta_i}\}$ *and* $\{g_i\in H_{M_{11},\delta_i}\}$ *such that for any* $i=1,2,...$ *the perturbed system*

$$\begin{cases}\dot{x}(t)=f_0(t,x,u(t))+f_i(t,x), t\in[t_0,\theta],\\ \dot{y}(t)=g_0(t,y,v(t))+g_i(t,y), t\in[\theta,t_1]\end{cases}$$

with the conditions

$$x(t_0)=x_0, y(\theta)=Q(\theta,x(\theta)), y(t_1)\in B_{y_1,\varepsilon_i} \tag{1.7}$$

is controllable i.e. there exists admissible element $w_i=(t_{0i},\theta_i,t_{1i},u_i,v_i)$.
Then the system (1.1) is controllable with the conditions (1.2)-(1.4). Here $K_{11}\subset O$ *and* $M_{11}\subset Y$ *are compact sets, containing some neighborhoods of* K_1 *and* M_1*, respectively.*

Theorem 1.4. *Let the conditions 1.5, 1.6 hold and let there exist sequences* $\{\varepsilon_i\}\to 0, \{\delta_i\}\to 0, \{f_i\in V^u_{K_{11},\delta_i}\}$ *and* $\{g_i\in H^v_{M_{11},\delta_i}\}$ *such that for any* $i=1,2,...$ *the perturbed system*

$$\begin{cases}\dot{x}(t)=f_0(t,x,u(t))+f_i(t,x,u(t)), t\in[t_0,\theta],\\ \dot{y}(t)=g_0(t,y,v(t))+g_i(t,y,v(t)), t\in[\theta,t_1]\end{cases}$$

with the conditions (1.7) is controllable. Then system (1.1) is controllable with conditions (1.2)-(1.4).

Finally, we note that Theorems, analogous to Theorems 1.1-1.4 are given in [1] for ordinary and delay differential equations. Optimal control problems for various classes of the two-stage and multi-stage systems are investigated in [2-17].

2. Auxiliary assertions

Theorem 2.1([1], p.101; [18], p.108). *Let* $\tilde{w}=(\tilde{t}_0,\tilde{\theta},\tilde{t}_1,\tilde{u}(\cdot),\tilde{v}(\cdot))\in W$ *be a given element and let* $\Phi(\tilde{w})$ *be the corresponding solution. For arbitrary* $\varepsilon>0$ *there exists a number* $\delta=\delta(\varepsilon)>0$ *such that for any* $f\in V_{\tilde{K}_1,\delta}$ *and* $g\in H_{\tilde{M}_1,\delta}$ *the perturbed two-stage system*

$$\begin{cases}\dot{x}(t)=f_0(t,x,\tilde{u}(t))+f(t,x), t\in[\tilde{t}_0,\tilde{\theta}],\\ \dot{y}(t)=g_0(t,y,\tilde{v}(t))+g(t,y), t\in[\tilde{\theta},\tilde{t}_1]\end{cases}$$

with the conditions

$$x(\tilde{t}_0)=x_0, y(\tilde{\theta})=Q(\tilde{\theta},x(\tilde{\theta}))$$

has the solution

$$\Phi(\tilde{w};f,g)=\{x(t;\tilde{w},f,g)\in\tilde{K}_1, t\in[\tilde{t}_0,\tilde{\theta}]; y(t;\tilde{w},f,g)\in\tilde{M}_1, t\in[\tilde{\theta},\tilde{t}_1]\}$$

and the following inequalities

$$|x(t;\tilde w)-x(t;\tilde w,f,g)|\le\varepsilon, t\in[\tilde t_0,\tilde\theta]; |y(t;\tilde w)-y(t;\tilde w,f,g)|\le\varepsilon, t\in[\tilde\theta,\tilde t_1]$$

hold.Here $\tilde K_1\subset O$ *and* $\tilde M_1\subset Y$ *are compact sets, containing some neighborhoods of* $\{x(t;\tilde w):t\in[\tilde t_0,\tilde\theta]\}$ *and* $\{y(t;\tilde w):t\in[\tilde\theta,\tilde t_1]\}$*, respectively.*

Theorem 2.2([1], p.101; [18], p.108). *Let the condition 1.5 hold. Then for arbitrary* $\varepsilon>0$ *there exists a number* $\delta=\delta(\varepsilon)>0$ *such that for any* $w\in W, f\in V_{K_{11},\delta}$ *and* $g\in H_{M_{11},\delta}$ *the perturbed two-stage system*

$$\begin{cases}\dot x(t)=f_0(t,x,u(t))+f(t,x), t\in[t_0,\theta],\\ \dot y(t)=g_0(t,y,v(t))+g(t,y), t\in[\theta,t_1]\end{cases}$$

with the conditions

$$x(t_0)=x_0, y(\theta)=Q(\theta,x(\theta))$$

has the solution

$$\hat\Phi(w;f,g)=\{\hat x(t;w,f,g)\in K_{11}, t\in I_1; \hat y(t;w,f,g)\in M_{11}, t\in I_2\}$$

and the following inequalities

$$|\hat x(t;w)-\hat x(t;w,f,g)|\le\varepsilon, t\in I_1; |\hat y(t;w)-\hat y(t;w,f,g)|\le\varepsilon, t\in I_2$$

hold.

Lemma 2.1 ([19], p.86). *Let* $x(t)\in O, t\in I_1$ *be a continuous function and let a sequence* $\{f_i\in V_{K,C}\}$ *satisfy the condition*

$$\lim_{i\to\infty}\sup\Big\{\Big|\int_{s_1}^{s_2}f_i(t,x)dt\Big| : s_1,s_2\in I_1, x\in K\Big\}=0.$$

Then

$$\lim_{i\to\infty}\sup\Big\{\Big|\int_{s_1}^{s_2}f_i(t,x(t))dt\Big| : s_1,s_2\in I_1\Big\}=0.$$

Here $K\subset O$ *is a compact set containing some neighborhood of* K.

Let $p(t,u)\in R_x^n$ be a given function, defined on $I_1\times U$ and satisfying the following conditions: for almost all $t\in I_1$ the function $p(t,\cdot)\to R_x^n$ is continuous; for each $u\in U$ the function $p(\cdot,u):I_1\to R_x^n$ is measurable.

Theorem 2.3([20], p.257). *Let the set*

$$P(t)=\{p(t,u):u\in U\}$$

be convex and

$$p_i(\cdot)\in L_1(I_1,R_x^n); p_i(t)\in P(t)\ \ a.e.\ \ on\ \ I_1, i=1,2,....$$

moreover,

$$\lim_{i\to\infty}p_i(t)=p(t)\quad weakly\ on\ \ I_1.$$

Then

$$p(t) \in P(t) \;\; a.e. \;\; on \;\; I_1$$

and there exists a measurable function $u(t) \in U, t \in I_1$ *such that*

$$p(t, u(t)) = p(t) \;\; a.e. \;\; on \;\; I_1.$$

3. Proof of Theorem 1.1

Let $\varepsilon_0 > 0$ be so small that

$$K_{\varepsilon_0} = \{x \in R_x^n : \exists \hat{x} \in K_0, |x - \hat{x}| \leq \varepsilon_0\} \subset intK_{01}$$

and

$$M_{\varepsilon_0} = \{y \in R_y^m : \exists \hat{y} \in M_0, |y - \hat{y}| \leq \varepsilon_0\} \subset intM_{01}.$$

According to Theorem 2.1 for any $\varepsilon \in (0, \varepsilon_0]$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that for any $f \in V_{K_{01},\delta}$ and $g \in H_{M_{01},\delta}$ the perturbed two-stage system

$$\begin{cases} \dot{x}(t) = f_0(t, x, u_0(t)) + f(t, x), t \in [t_{00}, \theta_0], \\ \dot{y}(t) = g_0(t, y, v_0(t)) + g(t, y), t \in [\theta_0, t_{10}] \end{cases}$$

with the conditions

$$x(t_{00}) = x_0, y(\theta_0) = Q(\theta_0, x(\theta_0))$$

has the solution

$$\Phi(w_0; f, g) = \{x(t; w_0, f, g), t \in [t_{00}, \theta_0]; y(t; w_0, f, g), t \in [\theta_0, t_{10}]\}$$

and the following inequalities

$$|x(t; w_0) - x(t; w_0, f, g)| \leq \varepsilon, t \in [t_{00}, \theta_0]; |y(t; w_0) - y(t; w_0, f, g)| \leq \varepsilon, t \in [\theta_0, t_{10}]$$

hold.

Consequently, the element w_0 is admissible for system (1.5) with conditions (1.6) for any $f \in V_{K_{01},\delta}$ and $g \in H_{M_{01},\delta}$.

Remark 2.1. Theorem 1.2 is a simply corollary of Theorem 1.1, since for any $u(\cdot) \in \Omega$ and $v(\cdot) \in \Delta$ we have

$$\sup\left\{\left|\int_{s_1}^{s_2} f(t, x, u(t))dt\right| : s_1, s_2 \in I_1, x \in K\right\} \leq \int_{I_1} \sup_{(x,u) \in K \times U} |f(t, x, u)|dt,$$

$$\sup\left\{\left|\int_{s_1}^{s_2} g(t, y, v(t))dt\right| : s_1, s_2 \in I_2, y \in M\right\} \leq \int_{I_2} \sup_{(y,v) \in M \times V} |g(t, y, v)|dt.$$

4. Proof of Theorem 1.3

Let $\varepsilon_0 > 0$ be so small that

$$K_{1,\varepsilon_0} = \{x \in R_x^n : \exists \hat{x} \in K_1, |x - \hat{x}| \leq \varepsilon_0\} \subset intK_{11}$$

and

$$M_{1,\varepsilon_0}=\{y\in R^m_y:\exists\hat{y}\in M_1,|y-\hat{y}|\le\varepsilon_0\}\subset intM_{11}.$$

It is clear that there exists a subsequence $\{\varepsilon_{i_1}\}\subset\{\varepsilon_1,\varepsilon_2,...\}$ such that $\varepsilon_{i_1}\in(0,\varepsilon_0], i=1,2,....$ On the basis of Theorem 2.2 for each ε_{i_1} there exists $\delta_{i_1}\in\{\delta_1,\delta_2,...\}$ such that for $w_{i_1}=(t_{0i_1},\theta_{i_1},t_{1,i_1},u_{i_1},v_{i_1}), f_{i_1}$ and g_{i_1} we have

$$|x(t;w_{i_1})-x(t;w_{i_1},f_{i_1},g_{i_1})|\le\varepsilon_{i_1}, t\in I_1 \tag{4.1}$$

and

$$|y(t;w_{i_1})-y(t;w_{i_1},f_{i_1},g_{i_1})|\le\varepsilon_{i_1}, t\in I_2. \tag{4.2}$$

Thus,

$$x(t;w_{i_1},f_{i_1},g_{i_1})\in K_{1,\varepsilon_0}, t\in I_1$$

and

$$y(t;w_{i_1},f_{i_1},g_{i_1})\in M_{1,\varepsilon_0}, t\in I_2.$$

The sequences $\{x(t;w_{i_1})\}$ and $\{y(t;w_{i_1})\}$ are uniformly bounded and equicontinuous, since

$$x(t;w_{i_1})\in K_1, t\in I_1; y(t;w_{i_1})\in M_1, t\in I_2$$

and

$$|\dot{x}(t;w_{i_1})|\le|f_0(t,x(t;w_{i_1}),u_{i_1}(t))|\le m_{K_1}(t)=m_{f_0,K_1}(t), t\in I_1,$$
$$|\dot{y}(t;w_{i_1})|\le|g_0(t,y(t;w_{i_1}),v_{i_1}(t))|\le m_{M_1}(t)=m_{g_0,M_1}(t), t\in I_2.$$

By the Arzela-Ascoli lemma from sequences $\{x(t;w_{i_1})\}$ and $\{y(t;w_{i_1})\}$ we can extract uniformly convergent subsequences. Without loss of generality, we assume that

$$\lim_{i\to\infty}x(t;w_{i_1})=x_0(t) \text{ uniformly in } I_1, \tag{4.3}$$

$$\lim_{i\to\infty}y(t;w_{i_1})=y_0(t) \text{ uniformly in } I_2; \tag{4.4}$$

$$\lim_{i\to\infty}t_{0i_1}=t_{00}, \lim_{i\to\infty}\theta_{i_1}=\theta_0, \lim_{i\to\infty}t_{1i_1}=t_{10}.$$

On the basis of (4.1)-(4.4) we obtain

$$\lim_{i\to\infty}x_{i_1}(t)=x_0(t) \text{ uniformly in } I_1, \lim_{i\to\infty}y_{i_1}(t)=y_0(t) \text{ uniformly in } I_2,$$

where

$$x_{i_1}(t)=x(t;w_{i_1},f_{i_1},g_{i_1}), y_{i_1}(t)=y(t;w_{i_1},g_{i_1}),y_{i_1}).$$

Obviously,

$$x_{i_1}(t_{0i_1})=x_0, y_{i_1}(\theta_{i_1})=Q(\theta_{i_1},x_{i_1}(\theta_{i_1})), y_{i_1}(t_{i_1})\in B_{y_1,\varepsilon_{i_1}},$$

therefore

$$x_0(t_{00})=x_0, y_0(\theta_0)=Q(\theta_0,x_0(\theta_0)), y_0(t_{10})=y_1. \tag{4.5}$$

Further,

$$x_{i_1}(t)=x_0+\int_{t_{0i_1}}^t[f_0(s,x_{i_1}(s),u_{i_1}(s))+f_{i_1}(s,x_{i_1}(s))]ds=x_0+\int_{t_{0i_1}}^t p_i(s)ds+\alpha_i(t)$$

$$+\beta_i(t)+\gamma_i(t), \tag{4.6}$$

where

$$p_i(s)=f_0(s,x_0(s),u_{i_1}(s)), \alpha_i(t)=\int_{t_{0i_1}}^{t} f_{i_1}(s,x_0(s))ds,$$

$$\beta_i(t)=\int_{t_{0i_1}}^{t}[f_0(s,x_{i_1}(s),u_{i_1}(s))-p_i(s)]ds, \gamma_i(t)=\int_{t_{0i_1}}^{t}[f_{i_1}(s,x_{i_1}(s))-f_{i_1}(s,x_0(s))]ds.$$

It is not difficult to see that

$$|p_i(s)|\le m_{K_{i_1}}(t), i=1,2,..., |\alpha_i(t)|\le \sup\Big\{\Big|\int_{s_1}^{s_2} f_{i_1}(t,x_0(t))dt\Big| : s_1,s_2\in I_1\Big\},$$

$$|\beta_i(t)|\le \max_{t\in I_1}\Big|x_{i_1}(t)-x_0(t)\Big|\int_{I_1} L_{K_{11}}(s)ds,$$

$$|\gamma_i(t)|\le \max_{t\in I_1}\Big|x_{i_1}(t)-x_0(t)\Big|\int_{I_1} L_{f_{i_1},K_{11}}(s)ds\le C\Big|x_{i_1}(t)-x_0(t)\Big|.$$

Without loss of generality,we assume that

$$\lim_{i\to\infty} p_i(s)=p(s) \quad \text{weakly on } \; I_1$$

([20], p.239). Moreover, we have

$$\lim_{i\to\infty}\alpha_i(t)=0, \lim_{i\to\infty}\beta_i(t)=0, \lim_{i\to\infty}\gamma_i(t)=0$$

(see Lemma 2.1, 4.3 and 4.4). From (4.6) it follows

$$x_0(t)=x_0+\int_{t_{00}}^{t} p(s)ds, t\in[t_{00},\theta_0].$$

Obviously,

$$p_i(s)\in P(s)=f_{i_1}(s,x_0(s),U), s\in I_1.$$

From Theorem 2.3 follow the inclusion $p(s)\in P(s)$ and existence of such a function $u_0(\cdot)\in\Omega$ that

$$p(s)=f_0(s,x_0(s),u_0(s)), \; a.e. \; on \; I_1.$$

Thus,

$$x_0(t)=x_0+\int_{t_{00}}^{t} f_0(s,x_0(s),u_0(s))ds, t\in[t_{00},\theta_0].$$

In a similar way, taking into account convexity of the set $g_0(t,y,V)$, one can prove

$$y_0(t)=Q(\theta_0,x_0(\theta_0))+\int_{\theta_0}^{t} g_0(s,y_0(s),v_0(s))ds, t\in[\theta_0,t_{10}], v_0(\cdot)\in\Delta.$$

Consequently, the element $w_0=(t_{00},\theta_0,t_{10},u_0(\cdot),v_0(\cdot))$ is admissible (see (4.5)).

Remark 4.1. Theorem 1.4 is proved analogously to Theorem 1.3.

REFERENCES

1. Tadumadze T.A. Some Topics of Qualitative theory of Optimal Control. (Russian) *Tbilisi State Univ., Tbilisi*, 1983.

2. Velichenko V.V. Optimal control of composite systems. (Russian) *Dokl. Akad. Nauk SSSR*, **176** (1967), 754-756.

3. Medvedev V.A., Rozova V.N. Optimal control of stepped systems. (Russian) *Avtom. Telemekh.* **3** (1972), 15-23;

4. Warga J. Optimal Control of Differential and Functional Equations. (Russian) *Nauka, Moskow*, 1977.

5. Gogodze I.K. About optimal control of systems, which changes his structure on contiguous intervals of time. (Russian) *Trudy Inst. Sistem Upravleniya Akad. Nauk Gruzin. SSR*, **16** (1980), 5-9.

6. Kharatishvili G.L. The maximum principle in optimal problems with switching. (Russian) *Trudy Inst. Sistem Upravleniya Akad. Nauk Gruzin. SSR*, **19**, 1 (1980), 5-17.

7. Boltianskii V.G. The problem of optimization with change of phase space. (Russian) *Differentsial'nye Uravnenia*, **19**, 3 (1983), 518-521.

8. Nikol'skii M.S. and Akhmadaliev A.A linear problem of terminal control of composite systems. (Russian) *Vestnik Moskov. Univ. Ser.* ***XV*** *Vychisl. Mat. Kibernet.*, **1** (1984), 71-77.

9. Kharatishvili G.L., Tadumadze T.A. The problem of optimal control for nonlinear systems with variable structure, delays and piecewise-continuous prehistory. *Mem. Diff. Eq. Math. Phys.*, **11** (1997), 67-88 .

10. Tadumadze T. and Arsenashvili A. Optimal control of delayed variable structure systems with discontinuous initial condition. *Appl. Comput. Math.*, **4**, 1 (2005), 48-53.

11. Arsenashvili A., Ramishvili I., Tadumadze T. Necessary conditions of optimality for neutral variable structure optimal problems with discontinuous initial condition. *Mem. Diff. Eq. Math. Phys.*, **39** (2006), 1-34.

12. Tadumadze T., Arsenashvili A., Ramishvili I. Optimization of two-stage neutral controlled system with continuous initial condition. *Bull. Georg. Natl. Acad. Sci.*, **174**, 3 (2006), 396-398 .

13. Kharatishvili G.L., Tadumadze T.A. Variation formulas of solutions and optimal control problems for differential equations with retarded argument.*J. Math. Sci.(N.Y.)*, **104**, 1 (2007), 1-175.

14. Tadumadze T.A., Arsenashvili A.I. and Ramishvili I.V. Linearized maximum principle for neutral-type variable-structure optimal problems with delays in controls. *J. Math. Sci. (N.Y.)*, **148**, 1 (2008), 382-398.

15. Tadumadze T., Arsenashvili A. Optimization of a delay variable structure control system with mixed intermediate condition. *Bull. Georg. Natl. Acad. Sci. (N.S.)*, **2**, 3 (2008), 22-26 .

16. Tadumadze T. and Arsenashvili A. Optimal control of variable structure systems with delays and mixed intermediate condition. *Funct. Differ. Equ.*, **17**, 3-4 (2010), 295-317.

17. Mansimov K., Nasiyati M. Necessary optimality conditions of second order in discrete two-parameter stepwise control problems. *Semin. I. Vekua Inst. Appl. Math. Rep.,* **36-37** (2010-1011), 51-65.

18. Tadumadze T.A., Avalishvili N.M. Regular perturbations in optimal problems with variable structure. (Russian) *Optimal problems in systems with variable structure*, *Tbilisi State Univ., I.N. Vekua Institute of Applied Mathematics, Tbilisi*, 1985, 100-154.

19. Gamkrelidze R. V. Optimal Control. (Russian) *Tbilisi State Univ., Tbilisi*, 1977.

20. Ekeland I. and Temam R. Convex Analysis and Variational Problems. (Russian) *Mir, Moskow*, 1976.

Received 15.07.2012; accepted 19.10.2012.

Author's address:

T. Tadumadze
I. Vekua Institute of Applied Mathematics of

Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia

Department of Mathematics of
Iv. Javakhishvili Tbilisi State University
11, University St., Tbilisi 0186
Georgia
E-mail: tamaztad@yahoo.com,
tamaz.tadumadze@tsu.ge