Seminar of I. Vekua Institute of Applied Mathematics REPORTS, Vol. 38, 2012

ON ONE PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURES FOR A SQUARE WHICH IS WEAKENED BY A HOLE AND BY CUTTINGS AT VERTICES

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Abstract. In the present work we consider the problem of statics of the linear theory of elastic mixtures for a square which is weakened by a hole and by cuttings at vertices about of finding an equally strong contour. The hole and cutting boundaries are assumed to be free from external forces, and to the remaing part of the square boundary are applied the same absolutely rigid punches, subjected to the action of external normal contractive forces with the given principal vectors.

Relying on the analogous to Kolosov-Muskhelishvilis formulas, in the linear theory of elastic mixtures, the problem reduces to a mixed problem of the theory of analytic functions (the Keldysh-Sedov problem), and the solution of the latter allows us to construct complex potentials and equations of an unknown contour efficiently (in analytical form). The analysis of the obtained results is carried out and the formula of tangential normal stress vector is derived.

Keywords and phrases: Equally strong contour, elastic mixture, generalized Kolosov-Muskhelishvili representation, Keldysh-Sedov problem.

AMS subject classification (2010): 74B05.

Introduction

The problems of the plane theory of elasticity for infinite domains weakened by equally strong holes have been studied in [1], [8] and also by many other authors. The same problem for simple and doubly-connected domains with partially unknown boundaries are investigated in [2], [9] etc. The mixed boundary value problems of the plane theory of elasticity for domains with partially unknown boundaries have been studied by R. Bantsuri [3]. Analogous problem in the case of the plane theory of elastic mixtures is considered in [16].

In [14], using the method, suggested by R. Banstsuri in [4], the author gives a solution of the mixed problem of the plane theory of elasticity for a finite multiply connected domain with a partially unknown boundary having the axis of symmetry. Analogous problem in the case of the plane theory of elastic mixtures has been studied in [17]. In the work of R. Bantsuri and G. Kapanadze [5] the problem, of statics of the plane theory of elasticity, of finding an equally strong contour for a square which is weakened by a hole and by cuttings at vertices are considered.

In the present work, in the case of the plane theory of elastic mixtures we study the problem, analogous to that solved in [5]. For the solution of the problem the use will be made of the generalized Kolosov-Muskhelishvili's formula [17] and the method, developed in [5].

1. Some auxiliary formulas and operators

The homogeneous equation of statics of the theory of elastic mixtures in a complex form looks as follows [7]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + \mathcal{K} \frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0, \tag{1.1}$$

where $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$, $\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big)$, $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big)$, $U = (u_1 + iu_2, u_3 + iu_4)^T$, $u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$ are partial displacements

$$\mathcal{K} = -\frac{1}{2} e m^{-1}, \quad e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \quad m^{-1} = \frac{1}{\Delta_0} \begin{bmatrix} m_3 & -m_2 \\ -m_2 & m_1 \end{bmatrix},$$
$$\Delta_0 = m_1 m_3 - m_2^2, \quad m_k = e_k + \frac{1}{2} e_{3+k}, \quad e_1 = a_2/d_2,$$
$$e_2 = -c/d_2, \quad e_3 = a_1/d_2, \quad d_2 = a_1 a_2 - c^2, \quad a_1 = \mu_1 - \lambda_5, \quad a_2 = \mu_2 - \lambda_5,$$
$$a_3 = \mu_3 + \lambda_5, \quad e_1 + e_4 = b/d_1, \quad e_2 + e_5 = -c_0/d_1, \quad e_3 + e_6 = a/d_1,$$
$$a = a_1 + b_1, \quad b = a_2 + b_2, \quad c_0 = c + d, \quad d_1 = ab - c_0^2,$$
$$b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2\rho_2/\rho, \quad b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2\rho_1/\rho, \quad \alpha_2 = \lambda_3 - \lambda_4,$$
$$\rho = \rho_1 + \rho_2, \quad d = \mu_2 + \lambda_3 - \lambda_5 - \alpha_2\rho_1/\rho \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2\rho_2/\rho.$$

Here μ_1, μ_2, μ_3, λ_p, $p = \overline{1,5}$ are elasticity modules, characterizing mechanical properties of a mixture, ρ_1 and ρ_2 are its particular densities. The elastic constants μ_1, μ_2, μ_3, λ_p $p = \overline{1,5}$ and particular densities ρ_1 and ρ_2 will be assumed to satisfy the conditions of inequality [13].

In [6] M. Bashaleishvili obtained the following representations:

$$U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} = m\varphi(z) + \frac{1}{2} e \overline{\varphi'(z)} + \overline{\psi(z)} \tag{1.2}$$

$$TU = \begin{pmatrix} (Tu)_2 - i(Tu)_1 \\ (Tu)_4 - i(Tu)_3 \end{pmatrix}$$
$$= \frac{\partial}{\partial s(x)} [(A - 2E)\varphi(z) + Bz\overline{\varphi(z)'} + 2\mu\overline{\psi(z)}], \tag{1.3}$$

where $\varphi(z) = (\varphi_1, \varphi_2)^T$ and $\psi(z) = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions:

$$A = 2\mu m, \quad \mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}, \quad B = \mu e, \quad m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\frac{\partial}{\partial s(x)} = -n_2 \frac{\partial}{\partial x_1} + n_1 \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial n(x)} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2}, \quad n = (n_1, n_2)^T$$

are the unit vectors of the other normal, $(Tu)_p$, $p = \overline{1,4}$, the stress components [6]

$$(Tu)_1 = r'_{11} n_1 + r'_{21} n_2, \quad (Tu)_2 = r'_{12} n_1 + r'_{22} n_2,$$
$$(Tu)_3 = r''_{11} n_1 + r''_{21} n_2, \quad (Tu)_4 = r''_{12} n_1 + r''_{22} n_2,$$

Consider the following vectors [16] or [17]

$$\overset{(1)}{\tau}=\begin{pmatrix}r'_{11}\\ r''_{11}\end{pmatrix}=\begin{bmatrix}a & c_0\\ c_0 & b\end{bmatrix}\begin{pmatrix}\theta'\\ \theta''\end{pmatrix}-2\frac{\partial}{\partial x_2}\mu\begin{pmatrix}u_2\\ u_4\end{pmatrix},$$
$$\overset{(2)}{\tau}=\begin{pmatrix}r'_{22}\\ r''_{22}\end{pmatrix}=\begin{bmatrix}a & c_0\\ c_0 & b\end{bmatrix}\begin{pmatrix}\theta'\\ \theta''\end{pmatrix}-2\frac{\partial}{\partial x_1}\mu\begin{pmatrix}u_1\\ u_3\end{pmatrix}, \tag{1.4}$$

$$\overset{(1)}{\eta}=\begin{pmatrix}r'_{21}\\ r''_{21}\end{pmatrix}=-\begin{bmatrix}a_1 & c\\ c & a_2\end{bmatrix}\begin{pmatrix}\omega'\\ \omega''\end{pmatrix}+2\frac{\partial}{\partial x_1}\mu\begin{pmatrix}u_2\\ u_4\end{pmatrix},$$
$$\overset{(2)}{\eta}=\begin{pmatrix}r'_{12}\\ r''_{12}\end{pmatrix}=\begin{bmatrix}a_1 & c\\ c & a_2\end{bmatrix}\begin{pmatrix}\omega'\\ \omega''\end{pmatrix}+2\frac{\partial}{\partial x_2}\mu\begin{pmatrix}u_1\\ u_3\end{pmatrix}, \tag{1.5}$$

$$\theta'=\operatorname{div}u',\quad \theta''=\operatorname{div}u'',\quad \omega'=\operatorname{rot}u',\quad \omega\operatorname{rot}u''.$$

Let $(\mathbf{n}$,S) be the right rectangular system, where S and n are respectively, the tangent and the normal of the curve L at the point $t=t_1+it_2$. Assume that $n=(n_1,n_2)^T=(\cos\alpha,\sin\alpha)^T$ and $S^0=(-n_2,n_1)^T=(-sin\alpha,\cos\alpha)^T$, where α is the angle of inclination of the normal n to the ox_1-axis.

Introduce the vectors

$$U_n=(u_1n_1+u_2n_2;u_3n_1+u_4n_2)^T,\ U_s=(u_2n_1-u_1n_2;u_4n_1-u_3n_2)^T; \tag{1.6}$$

$$\sigma_n=\begin{pmatrix}(Tu)_1n_1+(Tu)_2n_2\\ (Tu)_3n_1+(Tu)_4n_2\end{pmatrix},\quad \sigma_s=\begin{pmatrix}(Tu)_2n_1-(Tu)_1n_2\\ (Tu)_4n_1-(Tu)_3n_2\end{pmatrix}, \tag{1.7}$$

$$\sigma_t=\begin{pmatrix}[r'_{21}n_1-r'_{11}n_2, & r'_{22}n_1-r'_{12}n_2]^TS^0\\ [r''_{21}n_1-r''_{11}n_2, & r''_{22}n_1-r''_{12}n_2]^TS^0\end{pmatrix}. \tag{1.8}$$

Let us call (1.8) vector the tangential normal stress vector in the linear theory of elastic mixture.

After elementary calculations we obtain

$$\sigma_n=\overset{(1)}{\tau}\cos^2\alpha+\overset{(2)}{\tau}\sin^2\alpha+\eta\cos\alpha\sin\alpha,$$
$$\sigma_t=\overset{(1)}{\tau}\sin^2\alpha+\overset{(2)}{\tau}\cos^2\alpha-\eta\cos\alpha\sin\alpha,$$
$$\sigma_s=\frac{1}{2}(\overset{(2)}{\tau}-\overset{(1)}{\tau})\sin 2\alpha+\frac{1}{2}\eta\cos 2\alpha-\frac{1}{2}\varepsilon^*$$

where $\eta=\overset{(1)}{\eta}+\overset{(2)}{\eta}$, $\varepsilon^*=\overset{(1)}{\eta}-\overset{(2)}{\eta}$.

Direct calculations allow us to check on L [16]

$$\sigma_n+\sigma_t=\tau=\overset{(1)}{\tau}+\overset{(2)}{\tau}=2(2E-A-B)\operatorname{Re}\varphi'(t); \tag{1.9}$$

$$\sigma_n+2\mu\Big(\frac{\partial Us}{\partial s}+\frac{U_n}{\rho_0}\Big)+i\Big[\sigma_s-2\mu\Big(\frac{\partial U_n}{\partial s}-\frac{U_s}{\rho_0}\Big)\Big]=2\varphi'(t), \tag{1.10}$$

$$[(A-2E)\varphi(t)+Bt\overline{\varphi'(t)}+2\mu\overline{\psi(t)}]_L=-i\int_L e^{i\alpha}(\sigma_n+i\sigma_s)ds; \tag{1.11}$$

where $\det(2E - A - B) > 0$, $\frac{1}{\rho_0}$ is the curvature of L at the point t. Everywhere in the sequel it will be assumed that the components U_n and U_s are bounded [7].

Formulas (1.2), (1.3) and (1.9), (1.10) are analogous to those of Kolosov-Muskhelishvili in the linear theory of elastic mixture [12].

2. Statement of the problem and the method of its solving

Let an isotropic elastic mixture occupy on the plane $z = x_1 + ix_2$ a doubly-connected domain G, a square. The side lenght of square will be denoted by a^0.

Let to the boundary of the square which is weakened by an interior hole and cuttings at vertices be applied the same absolutely smooth rigid punches, subjected to the action of external normal contractive forces with the known principal vectors. The hole and cutting boundary is free from external forces.

We formulate the following problem: Find an elastic equilibrium of the square and analytic form of the hole and cutting contours under the condition that tangential normal stress vector, i. e. (1.8) vector, will take one and the same constant value $\sigma_t = K^0$, $K^0 = (K_1^0, K_2^0) = \text{const}$ on them.

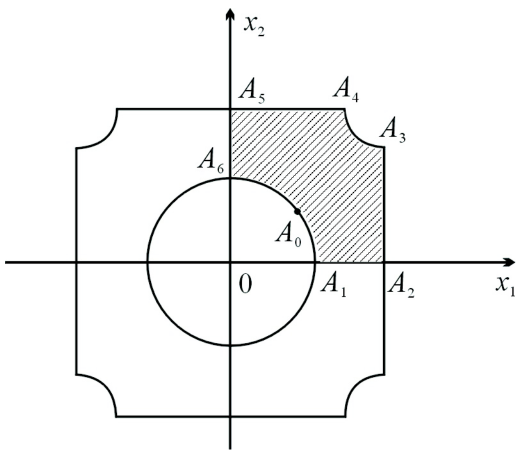

Figure 1:

In these conditions, we call the assemblage of hole and cutting boundaries an equally strong contour. Owing to the symmetry of the problem, we consider the shaded part of the square, i. e. the curvilinear polygon $A_0A_1A_2A_3A_4A_5A_6$ and denote it by D_0, where A_0 is the mid point of the arc A_6A_1 (a shaded in fig.1).

The boundary of the domain D_0 consists of rectilinear segments $L_1 = \cup L_1^{(j)}$, $L_1^{(j)} = A_jA_{j+1}$ $(j = 1, 2, 4, 5)$ and unknown arcs $L_0 = L_0^{(1)} \cup L_0^{(2)}$, $L_0^{(1)} = A_3A_4$, $L_0^{(2)} = A_6A_1$.

The boundary conditions of the problem are of the form $U_n = U^0 = \text{const}$ on $L_1^{(2)} \cup L_1^{(4)}$, and $U_n = 0$ on $L_1^{(1)} \cup L_1^{(5)}$, vector (1.7), is equal to zero on the entire boundary of the domain D_0, i.e. $\sigma_s = 0$ on $L = L_1 \cup L_0$.

Relying on the analogous Kolosov-Muskhelishvilis formulas (1.9)-(1.11), the above posed problem is reduced to finding two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in D^0

by the boundary conditions on L

$$\operatorname{Re}\varphi'(t) = H, \quad t \in L_0, \quad H = \frac{1}{2}(2E - A - B)^{-1}K^0, \tag{2.1}$$

$$\operatorname{Im}\varphi'(t) = 0, \quad t \in L_1, \tag{2.2}$$

$$\operatorname{Re} e^{-i\alpha(t)}[(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}] = C(t), \quad t \in L_1, \tag{2.3}$$

$$(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)} = B^{(j)}(t), \quad t \in L_0^{(j)}, \quad j = 1, 2; \tag{2.4}$$

where $\alpha(t)$ is the angle, made by the outer normal to the contour L_1 and the $0x_1$-axis,

$$C(t) = \operatorname{Re}\Big[-i \int_{A_1}^{t} \sigma_n(t_0) \exp i[\alpha(t_0) - \alpha(t)] dS_0 + \nu \exp(-i\alpha(t)) \Big], \ t \in L_1 \tag{2.5}$$

$$B^{(j)}(t) = -i \int_{A_1}^{t} \sigma_n(t_0) \exp(i\alpha(t_0)) dS_0 + \nu, \ \ t \in L_0^{(j)}, \ \ j = 1, 2, \tag{2.6}$$

$\nu = (\nu_1, \nu_2)^T$ is an arbitrary complex constant vector. It is easy to notice that $C(t)$ is a piecewise constant and $B^{(j)}$ is a constant vector-function.

Moreover, if $t \in L_1$, then we can write

$$\operatorname{Re} e^{-i\alpha(t)} t = \operatorname{Re} e^{-\alpha(t)} A(t) \tag{2.7}$$

where $A(t) = A_k$ for $t \in A_k A_{k+1}$.

In the sequel, the vector-function $\varphi(z)$ will be assumed to be continuous in a closed domain D_0, and $\varphi'(z)$ and $\psi(z)$ are continuously extendable on the boundary of the body D_0 except possibly of the points A_1, A_3, A_4, A_6 in the neighborhood of which they admit the estimate of the type

$$|\varphi_j'(z)|, \quad |\psi_j(z)| < M|z - A_k|^{-\delta_k}, \quad j = 1, 2, \tag{2.8}$$

where $0 < \delta_k < \frac{1}{2}$, $k = 1, 3, 4, 6$, $M = \text{const} > 0$.

The equalities (2.1)–(2.2) are in fact the Keldysh-Sedov problem for the domain D_0.

By virtue of the condition (2.8), the (2.1)–(2.2) problem has a unique solution [10] or [11], $\varphi'(z) = H$.

Consequently, leaving out of account the constant summand we get

$$\varphi(z) = Hz = \frac{1}{2}(2E - A - B)^{-1}K^0 z. \tag{2.9}$$

Here K^0 is to be defined in the course of solving the problem.

On the basis of formulas (1.11), (2.5), (2.6), (2.9) and putting $\nu = 0$, the boundary conditions (2.3), (2.4) and (2.7) yield

$$\begin{gathered}
\operatorname{Im}\left(\frac{1}{2}K^0t-2\mu\psi(t)\right)=0;\ \operatorname{Im}\left(\frac{1}{2}K^0t+2\mu\psi(t)\right)=0,\ t\in L_1^{(1)};\\
\operatorname{Re}\left(\frac{1}{2}K^0t-2\mu\psi(t)\right)=P;\ \operatorname{Re}\left(\frac{1}{2}K^0t+2\mu\psi(t)\right)=a^0K^0-P,\ t\in L_1^{(2)};\\
\operatorname{Re}\left(\frac{1}{2}K^0t-2\mu\psi(t)\right)=P;\ \operatorname{Im}\left(\frac{1}{2}K^0t+2\mu\psi(t)\right)=P,\ t\in L_0^{(1)};\\
\operatorname{Im}\left(\frac{1}{2}K^0t-2\mu\psi(t)\right)=a^0K^0-P;\ \operatorname{Im}\left(\frac{1}{2}K^0t+2\mu\psi(t)\right)=P,\ t\in L_1^{(4)};\\
\operatorname{Re}\left(\frac{1}{2}K^0t-2\mu\psi(t)\right)=0;\ \operatorname{Re}\left(\frac{1}{2}K^0t+2\mu\psi(t)\right)=0,\ t\in L_1^{(5)};\\
\operatorname{Re}\left(\frac{1}{2}K^0t-2\mu\psi(t)\right)=P;\ \operatorname{Im}\left(\frac{1}{2}K^0t+2\mu\psi(t)\right)=0,\ t\in L_0^{(2)};
\end{gathered} \tag{2.10}$$

where

$$P=\int_{L_1^{(j)}}\sigma_n dS,\quad j=1,2,4,5.$$

Let the function $z=\omega(\zeta)$, $\zeta=\xi_1+i\xi_2$ map conformally the upper half-plane ($\operatorname{Im}\zeta>0$) onto the domain D_0. By β_k we denote the preimages of the points A_k ($k=\overline{0,6}$) and assume that $\beta_3=-1$; $\beta_4=1$; $\beta_0=-\infty$. Moreover, owing to the symmetry, we may assume that $\beta_5=-\beta_2$; $\beta_6=-\beta_1$. Note that

$$-\infty<\beta_1<\beta_2<-1<1<-\beta_2<-\beta_1<+\infty.$$

Consider the vector-functions

$$\phi(\zeta)=-i\Big(\frac{1}{2}K^0\omega(\zeta)-2\mu\psi(\omega(\zeta))\Big); \tag{2.11}$$

$$\Psi(\zeta)=\frac{1}{2}K^0\omega(\zeta)+2\mu\psi(w(\zeta)). \tag{2.12}$$

Taking into (2.11) and (2.12), boundary conditions (2.10) take the form

$$\begin{gathered}
\operatorname{Im}\phi(\xi_1)=0,\quad \xi_1\in(-\infty;\beta_1)\cup(-\beta_2;\infty);\\
\operatorname{Re}\phi(\xi_1)=0;\quad \xi_1\in(\beta_1;\beta_2);\\
\operatorname{Im}\phi(\xi_1)=-P;\quad \xi_1\in(\beta_2;1),\\
\operatorname{Re}\phi(\xi_1)=a^0K^0-P,\quad \xi_1\in(1;-\beta_2);
\end{gathered} \tag{2.13}$$

$$\begin{gathered}
\operatorname{Im}\Psi(\xi_1)=0,\quad \xi_1\in(-\infty;\beta_2)\cup(-\beta_1;\infty);\\
\operatorname{Re}\Psi(\xi_1)=a^0K^0-P,\quad \xi_1\in(\beta_2;-1),\\
\operatorname{Im}\Psi(\xi_1)=P,\quad \xi_1\in(-1,-\beta_2),\\
\operatorname{Re}\Psi(\xi_1)=0,\quad \xi_1\in(-\beta_2,-\beta_1).
\end{gathered} \tag{2.14}$$

The above problems are the vector form of the Keldysh-Sedov problems [10], [11] for a half-plane $\operatorname{Im}\zeta>0$.

A solution of problems (2.13) and (2.14) can be represented as follows [10], [5]

$$\phi(\zeta)=\frac{\chi_1(\zeta)}{\pi i}\left[(a^0K^0-P)\int_1^{-\beta_2}\frac{d\xi_1}{\chi_1(\xi_1)(\xi_1-\zeta)}-iP\int_{\beta_2}^1\frac{d\xi_1}{\chi(\xi_1)(\xi_1-\zeta)}\right], \tag{2.15}$$

$$\Psi(\zeta)=\frac{\chi_2(\zeta)}{\pi i}\left[(a^0K^0-P)\int_{\beta_2}^{-1}\frac{d\xi_1}{\chi_2(\xi_1)(\xi_1-\zeta)}+iP\int_{-1}^{\beta_2}\frac{d\xi_1}{\chi_2(\xi_1)(\xi_1-\zeta)}\right], \tag{2.16}$$

where

$$\chi_1(\zeta)=\sqrt{(\zeta-\beta_1)(\zeta-\beta_2)(\zeta-1)(\zeta+\beta_2)},$$
$$\chi_2(\zeta)=\sqrt{(\zeta+\beta_1)(\zeta+\beta_2)(\zeta+1)(\zeta-\beta_2)}$$

Note that, under the $\chi_j(\zeta)$ sign we mean a branch whose decomposition near the point at infinity has the form

$$\chi_j(\zeta)=\zeta^2+\alpha_1^j\zeta+\alpha_2^{(j)}+\cdots,\quad j=1,2.$$

It is easy to show that

$$\chi_1(\xi_1)=\begin{cases}|\chi_1(\xi_1)|, & \xi_1\in(-\infty,\beta_1)\cup(\beta_2,1)\cup(-\beta_2,\infty)\\ -i|\chi_1(\xi_1)|, & \xi_1\in(\beta_1,\beta_2)\cup(1,-\beta_2);\end{cases} \tag{2.17}$$

$$\chi_2(\xi_1)=\begin{cases}|\chi_2(\xi_1)|, & \xi_1\in(-\infty,\beta_2)\cup(-1,-\beta_2)\cup(-\beta_1,\infty)\\ i|(\xi_1)|, & \xi_1\in(\beta_2,-1)\cup(-\beta_2,-\beta_1);\end{cases} \tag{2.18}$$

$$|\chi_1(\xi_1)|=|\chi_2(-\xi_1)|. \tag{2.19}$$

By virtue of (2.17)–(2.19) formulas (2.15) and (2.16) can be written as

$$\phi(\zeta)=g(\zeta),\quad \Psi(\zeta)=g(-\zeta),\quad \operatorname{Im}\zeta>0, \tag{2.20}$$

where $g=(g_1,g_2)^T$.

$$g(\zeta)=\frac{\chi_1(\zeta)}{\pi i}\left[(a^0K^0-P)\int_1^{-\beta_2}\frac{d\xi_1}{|\chi_1(\xi_1)|(\xi_1-\zeta)}-P\int_{\beta_2}^1\frac{d\xi_1}{|\chi_1(\zeta)|(\xi_1-\zeta)}\right]. \tag{2.21}$$

Now note that we will seek for a bounded at infinity solution of the problems (2.13) and (2.14). On the other hand, from (2.20) and (2.21) we conclude that, the necessary and sufficient condition for the existence of such a solution is of the form

$$(a^0K^0-P)\int_1^{-\beta_2}\frac{d\xi_1}{|\chi_1(\xi_1)|}-P\int_{\beta_2}^1\frac{d\xi_1}{|\chi_1(\xi_1)|}=0. \tag{2.22}$$

Having found the vector-function $\phi(\zeta)$ and $\Psi(\zeta)$, by virtue of (2.12) and (2.20) we can define the vector-functions $K^0\omega(\zeta)$ and $\psi(\omega(\zeta))$ by the formulas

$$K^0\omega(\zeta)=g(-\zeta)+ig(\zeta),\quad \psi(\omega(\zeta))=\frac{1}{4}\mu^{-1}[g(-\zeta)-ig(\zeta)]. \tag{2.23}$$

Let us now pass to finding analytical form of the unknown equally strong contour. Equations for the parts $L_0^{(1)}$ and $L_0^{(2)}$ of the unknown contour can be obtained from the image of the function $\omega(\zeta)$ for $\zeta = \xi_1^0 \in (-1, 1)$ and $\zeta = \xi_1^0 \in (-\infty, \beta_1) \cup (-\beta_1, \infty)$ respectively.

By the Sokhotskii-Plemelj formulas [11] and owing to (2.21) and (2.23), we find that the equations for the arcs $L_0^{(1)}$ and $L_0^{(2)}$ are given by the formulas respectively

$$\omega(\xi_1^0) = \frac{g_1(-\xi_1^0) + P_1 + i(g_1(\xi_1^0) + P_1)}{K_1^0} = \frac{g_2(-\xi_1^0) + i(g_2(\xi_1^0) + P_2)}{K_2^0}, \tag{2.24}$$

$$\omega(\xi_1^0) = \frac{g_1(-\xi_1^0) + i(g_1(\xi_1^0))}{K_1^0} = \frac{g_2(-\xi_1^0) + i(g_2(\xi_1^0))}{K_2^0}, \tag{2.25}$$

where

$$g(\xi_1^0) = (g_1, g_2)^T = \frac{\chi_1(\xi_1^0)}{\pi i}\left[(a^0K^0 - P)\int_1^{-\beta_2} \frac{d\xi_1}{|\chi_1(\xi_1)|(\xi_1 - \xi_1^0)} - P\int_{\beta_2}^1 \frac{d\xi_1}{|\chi_1(\xi_1^0)(\xi_1 - \xi_1^0)|}\right].$$

Revert now to the condition (2.22). Equality (2.22) yelds

$$K^0 = \frac{1}{a^0}P\left(1 + \frac{F_1}{F_2}\right), \tag{2.26}$$

where

$$F_1 = \int_{\beta_2}^1 \frac{d\xi_1}{|\chi_1(\xi_1)|}, \quad F_2 = \int_1^{-\beta_2} \frac{d\xi_1}{|\chi_1(\xi_1)|}. \tag{2.27}$$

It should be noted the integrals appearing in (2.27) and (2.21) are expressed in terms of elliptic integrals of the first and third kind [15].

Of special importance is the definition of parameters K_1^0, K_2^0, β_1 and β_2, involved in the above formulas. For defining above parameters we use the way and results, described in [5].

Refer now to formulas (2.26) and (2.27). the values F_1 and F_2 are the complete elliptic integrals of the first kind [15], namely [5]

$$F_1 = M^{-1}F\left(\frac{\pi}{2}/m_1^0\right); \quad F_2 = M^{-1}F\left(\frac{\pi}{2}/m_2^0\right),$$

where

$$M = \sqrt{2}[\beta_2(\beta_1 - 1)]^{-\frac{1}{2}}, \quad F(\frac{\pi}{2}/m^0) = \int_0^{\frac{\pi}{2}} (1 - m^0\sin^2\theta)^{-\frac{1}{2}}d\theta,$$

$$m_1^0 = \frac{2\beta_2(\beta_1 - 1)(\beta_1 + \beta_2)}{2\beta_2(\beta_1 - 1)}, \quad m_2^0 = \frac{(\beta_2 + 1)(\beta_1 - \beta_2)}{2\beta_2(\beta_1 - 1)}.$$

(of interest is the fact that $m_1^0 + m_2^0 = 1$ and $m_1^0 > m_2^0$).

Fixing the value of the parameter m_1^0 (and hence of parameter $m_2^0 = 1 - m_1^0$) for finding β_1 and β_2 we obtain the equality

$$\beta_2^2 + (1 - 2m_1^0)(\beta_1 - 1)\beta_2 - \beta_1 = 0 \tag{2.28}$$

the discriminant of the above equation(with respect to β_2) is of the form $D = (1 - 2m_1^0)^2(\beta_1 - 1)^2 + 4\beta_1$.

Introducing the notation $\sqrt{-\beta_1} = x$, from the condition $D \geq 0$, $x > 1$ we get

$$x \geq \frac{1 + 2\sqrt{m_1^0(1 - m_1^0)}}{2m_1^0 - 1} = l.$$

If we assume that $D > 0$, then to every value $x > l$, and hence $\beta_1 < -l^2$, according to (2.28), there correspond two values β_2, both satisfying the condition $\beta_2 < -1$, but this contradicts the condition of the uniqueness of the conformally mapping function $z = \omega(\zeta)$, and hence we should have $D = 0$ from which it follows that

$$\beta_1 = -\Big[\frac{1 + 2\sqrt{m_1^0(1 - m_1^0)}}{2m_1^0 - 1}\Big]^2; \quad \beta_2 = \frac{(2m_1^0 - 1)(\beta_1 - 1)}{2}. \tag{2.29}$$

Summing the obtained results, we conclude that for the fixed m_1^0 in the domain $(\frac{1}{2}, 1)$, from the table of complete elliptic integrals we can find F_1 and F_2, and using formulas (2.26) and (2.27) we define parameters $\mathcal{K}^0$, β_1, β_2 and the conformally mapping function $z = \omega(\zeta)$ formulas (2.24) and (2.25) which establishes analytical form of the unknown equally strong contour.

Direct calculations show that as m_1^0 increases, the length of the contour $L_0^{(1)}$ decreases, $L_0^{(2)}$ increases, and K_j^0 $j = 1, 2$, increases (see [5]).

In a particular case, for $m_1^0 = 0,75$ we have approximately [18]

$$\begin{gathered}
F_1 = 2,156; \quad F_2 = 1,686; \quad K_j^0 = \frac{2,28}{a^0}\,P_j, \quad j = 1,2; \\
\beta_1 = -13,7; \quad \beta_2 = -3,7; \quad g_j(0) = 0,743P_j, \quad j = 1,2; \\
\omega(0) = (0,764a^0; 0,764a^0); \\
g_j(-1) = 0,386P_j, \quad j = 1,2; \quad \omega(-1) = (a^0; 0,608a^0); \\
g_j(\infty) = g_j(-\infty) = 1,08P_j, \quad j = 1,2; \\
\omega(\infty) = \omega(-\infty) = (0,474a^0; 0,474a^0); \\
g_j(-\beta_1) = 1,451P_j; \quad \omega(\beta_1) = (0,636a^0; 0).
\end{gathered}$$

R E F E R E N C E S

1. Banicuk N.V. Optimization of Forms of Elastic Bodies. (Russian) *Nauka, Moscow,* 1980.

2. Bantsuri R.D., Isakhanov, R.S. A problem of elasticity theory for a domain with an unknown part of the boundary. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR*, **116**, 1 (1984), 45-48.

3. Bantsuri R.D. On one mixed problem of the plane theory of elasticity with a partially unknown boundary. *Proc. A. Razmadze Math. Inst.* **140** (2006), 9-16.

4. Bantsuri R.D. Solution of the mixed problem of plate bending for a multi-connected domain with partially unknown boundary in the presence of cyclic symmetry. *Proc. A. Razmadze Math. Inst.*, **145** (2007), 9-22.

5. Bantsuri R.D. and Kapanadze G. On one problem of finding an equally strong contour for a square which is weakened by a hole and by cuttings at vertices. *Proc. A. Razmadze Math. Inst.* **155** (2011), 9-16.

6. Basheleishvili M. Analogues of the Kolosov-Muskhelishvili general representation formulas and Cauchy-Riemann conditions in the theory of elastic mixtures. *Georgian Math. J.*, **4**, 3 (1997), 223-242.

7. Basheleishvili M., Svanadze K. A new method of solving the basic plane boundary value problems of statics of the elastic mixture theory. *Georgian Math. J.*, **8**, 3 (2001), 427-446.

8. Cherepanov G.P. Inverse problems of the plane theory of elasticity. *translated from Prikl. Mat. Meh.* **38**, 6 (1974), 963-979 *J. Appl. Math. Mech.*, **38**, 6 (1974), 915-931 (1975).

9. Kapanadze G.A. On one problem of the plane theory of elasticity with a partially unknown boundary. *Proc. A. Razmadze Math. Inst.*, **143** (2007), 61-71.

10. Keldysh M.V., Sedov, L.I. The effective solution of some boundary vaule problem for harmonic functions. (Russian) *DAN SSSR*, **16**, 1 (1937), 7-10.

11. Muskhelishvili N.I. Singular Integral Equations. (Russian) *Nauka, Moscow*, 1968.

12. Muskhelishvili N.I. Some Basic Problems of the Mathematical Theory of Elasticity. (Russian) *Nauka, Moscow*, 1966.

13. Natroshvili D.G., Dzhagmaidze A.Ya., Svanadze M.Zh. (Russian) Some Problems in the Linear Theory of Elastic Mixtures. *Tbilis. Gos. Univ., Tbilisi*, 1986.

14. Odishelidze N.T. Solution of the mixed problem of the plane theory of elasticity for a multiply connected domain with partially unknown boundary in the presence of axial symmetry. *Proc. A. Razmadze Math. Inst.*, **146** (2008), 97-112.

15. Prudnikov A.P. Brychkov, Yu.A., Marichev, O.I. Integrals and Series. (Russian) *Nauka, Moscow*, 1981.

16. Svanadze K. On one mixed problem of the plane theory of elastic mixture with a partially unknown boundary. *Proc. A. Razmadze Math. Inst.*, **150** (2009), 121-131.

17. Svanadze K. Solution of a mixed boundary value problem of the plane theory of elastic mixture for a multiply connected domain with a partially unknown boundary having the axis of symmetry. *Proc. A. Razmadze Math. Inst.*, **155** (2011), 73-90.

18. Handbook of Mathematical Functions. Edit. by M. Aramovich and I.A. Stegun. (Russian) *Nauka, Moscow*, 1979.

Received 15.05.2012; accepted 6.09.2012.

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