Seminar of I. Vekua Institute of Applied Mathematics
REPORTS, Vol. 38, 2012

COMPARISON THEOREMS AND SOME TWO-POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. Some two-point singular boundary value problems for second order linear differential equations are investigated.

Keywords and phrases: Comparison theorem, linear differential equation.

AMS subject classification (2010): 34K15.

1. Introduction

Consider the differential equations

$$u'' + p(t)\, u = 0 \tag{1.1}$$

and

$$v'' + q(t)\, v = 0, \tag{1.2}$$

where $p, q \in C((a,b); R)$. For these equations Sturm [1] proved a comparison theorem, which later was widely used in studying the boundary value problems and asymptotic behavior of solutions. For generalizations of Sturm's theorems see [2], and for singular case see [3-5].

2. Some auxiliary lemmas

Lemma 2.1. *Let* $a < t_0 < b$,

$$p, q \in C((a, t_0]; R_+) \tag{2.1}$$

and

$$\int_t^{t_0} (p(s) - q(s))ds \geq 0 \quad for \quad t \in (a, t_0]. \tag{2.2}$$

Let $v \in C^{(2)}((a, t_0]; [0, +\infty))$ *be a solution of equation* (1.2) *under conditions*

$$\lim_{t \to a_0+} v(t) = v(a_0+) = 0, \quad v'(t_0) = 0$$

and

$$\int_{a_0}^{t_0} \frac{ds}{v^2(s)} = +\infty,$$

where $a \leq a_0$, $v(t) > 0$ *for* $t \in (a_0, t_0]$ *and* $v(a_0) = 0$. *If* $u \in C^{(2)}((a_0, t_0]; R)$ *is a solution of equation* (1.1), *then at least one of the conditions*

1) *there exist* $t_* \in (a, t_0)$ *such that* $u(t_*) = 0$

or

2) $u'(t_0) \le 0$

is fulfilled.

Lemma 2.2. *Let* $a < t_0 < b$

$$p, q \in C([t_0, b); R_+) \tag{2.3}$$

and

$$\int_{t_0}^{t} (p(s) - q(s))ds \ge 0 \quad \textit{for} \quad t \in [t_0, b).$$

Let $v \in C^{(2)}([t_0 b); [0, +\infty))$ *be a solution of equation* (1.2) *under conditions*

$$v(b_0-) = \lim_{t \to b_0-} v(t) = 0, \quad v'(t_0) = 0$$

and

$$\int_{t_0}^{b_0} \frac{ds}{v^2(s)} = +\infty,$$

where $b_0 \le b$, $v(t) > 0$ *for* $t \in [t_0, b_0)$ *and* $v(b_0) = 0$. *If* $u \in C^{(2)}[t_0, b_0); R)$ *is a solution of equation* (1.1), *then at least one of the conditions*

1) *there exist* $t_* \in (t_0, b_0)$ *such that* $u(t_*) = 0$

or

2) $u'(t_0) \ge 0$

is fulfilled.

Remark 2.1. If in Lemmas 2.1 and 2.2 $p(t) \ge q(t)$ for $t \in (a, b)$, then conditions (2.1) and (2.3) are unnecessary.

3. Two-point boundary value problems

Consider the problems on the existence of solution of the equation

$$u'' + q(t)\, u = f(t), \tag{3.1}$$

where $q, f \in C((a, b); R)$, under conditions

$$u(a+) = 0, \quad u'(b-) = 0, \tag{3.2}$$

$$u'(a+) = 0, \quad u(b-) = 0 \tag{3.3}$$

and

$$u(a+) = u(b-) = 0. \tag{3.4}$$

Theorem 3.1. *Let* $q \in C((a, b]; R_+)$, $(t - a)q \in L([a, b])$ *and there exists a function* $p \in C((a, b]; R_+)$ *such that*

$$\int_{t}^{b} (p(s) - q(s))ds \ge 0 \quad \textit{for} \quad t \in (a, b),$$

and equation (1.1) *has a solution* $u : (a, b) \to (0, +\infty)$ *such that* $u'(b) > 0$. *Then problem* (3.1), (3.2) *has only one solution.*

Corollary 3.1. Let $q \in C((a,b];R_+)$, $(t-a)q \in L([a,b])$ and

$$\int_t^b p(s)ds \le \frac{b-t}{4(b-a)(t-a)} \quad \text{for} \quad t \in (a,b].$$

Then problem (3.1), (3.2) has only one solution.

Corollary 3.2. Let $q \in C((a,b];R_+)$, $(t-a)q \in L([a,b])$,

$$q(t) \le \frac{1}{4(t-a)^2} \quad \text{for} \quad t \in [a,b).$$

Then problem (3.1), (3.3) has only one solution.

Theorem 3.2. *Let $q \in C([a,b);R_+)$, $(b-t)q \in L([a,b])$ and there exists a function $p \in C([a,b];R_+)$ such that*

$$\int_a^t (p(s)-q(s))ds \ge 0 \quad \textit{for} \quad t \in [a,b),$$

and equation (1.1) has a solution $u : (a,b) \to (0,+\infty)$ such that $u'(a) < 0$. Then problem (3.1), (3.3) has only one solution.

Corollary 3.3. Let $q \in C([a,b);R_+)$, $(b-t)q \in L([a,b]$ and

$$\int_a^t q(s)ds \le \frac{t-a}{4(b-a)(b-t)} \quad \text{for} \quad t \in [a,b).$$

Then problem (3.1), (3.3) has only one solution.

Corollary 3.4. Let $q \in C([a,b);R)$, $(b-t)q \in L([a,b])$ and

$$q(t) \le \frac{1}{4(b-t)^2} \quad \text{for} \quad a \le t < b.$$

Then problem (3.1), (3.2) has only one solution.

Theorem 3.3 *Let $p;q \in C((a,b);R)$, $(t-a)(b-t)q \in L([a,b])$ and*

$$q(t) \le p(t) \quad \textit{for} \quad t \in (a,b). \tag{3.5}$$

If there exist $t_* \in (a,b)$ and solution $u \in C^{(2)}((a,b);(0,+\infty))$ of equation (1.1) such that $u'(t) > 0$ for $t \in (a,t_*]$ or $u'(t) < 0$ for $t \in [t_*,b)$, then problem (3.1), (3.4) has only one solution.

Corollary 3.5. Let $q \in C((a,b);R)$, $(t-a)(b-t)q \in L([a,b])$ and let (3.5) be fulfilled, where

$$p(t) = \begin{cases} \dfrac{1}{4(t-a)^2} & \text{for} \quad t \in \left(a, \frac{a+b}{2}\right], \\ \dfrac{1}{4(b-t)^2} & \text{for} \quad t \in \left[\frac{a+b}{2}, b\right). \end{cases}$$

Then problem (3.1), (3.4) has only one solution.

R E F E R E N C E S

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Received 25.07.2012; accepted 17.10.2012.

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