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ON THE EXISTENCE OF UNBOUNDED OSCILLATORY SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THIRD ORDER

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Abstract. The statements on the existence of unbounded oscillatory solutions are proved. It is also shown that non-oscillatory solutions vanish at infinity for linear ordinary differential equations of third order.

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Let us consider the linear ordinary differential equation of third order

$$u''' + p_1(t)u'' + p_2(t)u' + p_3(t)u = 0, \tag{1}$$

where $p_k : R_+ \to R$ $(k = 1, 2, 3)$ are continuous functions.

A nontrivial solution of equation (1) is called oscillatory if it has an infinite number of zeros, and non-oscillatory otherwise. In the present paper, when p_3 is non-negative, we prove the statements on the existence of unbounded oscillatory solutions, and also show that non-oscillatory solutions vanish at infinity.

We will first prove some auxiliary propositions.

Lemma 1. *Let* $\alpha \le 1$, *let the conditions*

$$\limsup_{t\to+\infty} t^{k\alpha}|p_k(t)| < +\infty \quad (k = 1, 2, 3) \tag{2}$$

be fulfilled and let equation (1) *have a solution, satisfying for some* $\mu \ge 0$ *the condition*

$$\limsup_{t\to+\infty} t^{-\mu}|u(t)| < +\infty. \tag{3}$$

Then

$$\limsup_{t\to+\infty} t^{-\mu+j\alpha}|u^{(j)}(t)| < +\infty \quad (j = 1, 2). \tag{4}$$

Proof. By (2) and (3) we can choose numbers $t_0 \ge 1$ and $c > 1$ such that

$$t^{k\alpha}|p_k(t)| < c \quad (k = 1, 2, 3) \ \text{ for } \ t \ge t_0, \tag{5}$$

$$t^{-\mu}|u(t)| < c \ \text{ for } \ t \ge t_0. \tag{6}$$

Therefore

$$|u'''(t)| \le c\sum_{j=0}^{2} t^{(j-3)\alpha}|u^{(j)}(t)| \ \text{ for } \ t \ge t_0. \tag{7}$$

Assume that the lemma is not true, i.e.

$$\limsup_{t\to+\infty}\sum_{j=1}^{2} t^{-\mu+j\alpha}|u^{(j)}(t)| = +\infty.$$

Then there exist increasing sequences $(t_i)_{i=1}^{+\infty}$, $(M_i)_{i=1}^{+\infty}$ such that $t_1 > t_0$, $t_i \to +\infty$, $M_i \to +\infty$ as $i \to +\infty$ and

$$M_i = \sum_{j=1}^{2} t^{-\mu+j\alpha}|u^{(j)}(t)| = \max\left\{\sum_{j=1}^{2} t^{-\mu+j\alpha}|u^{(j)}(t)| :\ t_0 \le t \le t_i\right\}.$$

Thus we can assume that there exists $l \in \{1,2\}$ such that for any $i \in N$

$$t_i^{-\mu+l\alpha}|u^{(l)}(t_i)| \ge \frac{M_i}{2}\,.$$

Suppose first that $l = 2$ and $h > 0$ satisfies the inequalities

$$hc < \frac{1}{4}\,, \quad hc(1-h)^{\mu-3\alpha} < \frac{1}{4}\,.$$

Then by virtue of (7)

$$|u''(t)| \ge |u''(t_i)| - \int_t^{t_i} |u'''(s)|\,ds \ge \frac{M_i}{2}\,t_i^{\mu-2\alpha} - \int_t^{t_i} cM_i s^{\mu-3\alpha}\,ds$$

and therefore if $\mu - 3\alpha \ge 0$, then

$$|u''(t)| \ge \frac{M_i}{2}\,t_i^{\mu-2\alpha} - cM_i t_i^{\mu-3\alpha} h t_i^{\alpha} \ge \frac{M_i}{4}\,t_i^{\mu-2\alpha} \ \text{ for }\ t \in [t_i - ht_i^{\alpha}; t_i],$$

and if $\mu - 3\alpha < 0$, then

$$|u''(t)| \ge \frac{M_i}{2}\,t_i^{\mu-2\alpha} - cM_i t_i^{\mu-3\alpha}(1-h)^{\mu-3\alpha} h t_i^{\alpha} \ge \frac{M_i}{4}\,t_i^{\mu-2\alpha} \quad \text{for }\ t \in [t_i - ht_i^{\alpha}; t_i].$$

Let $s_0 = t_i - ht_i^{\alpha}$, $s_1 = t_i - \frac{ht_i^{\alpha}}{2}$, $s_2 = t_i$. Then there exists $\xi \in [s_0, s_2]$ such that

$$\frac{u(\xi)}{2} = \frac{u(s_0)}{(s_1-s_0)(s_2-s_0)} - \frac{u(s_1)}{(s_1-s_0)(s_2-s_1)} + \frac{u(s_2)}{(s_2-s_0)(s_2-s_1)}\,.$$

Hence by virtue of (6) we obtain

$$\frac{M_i}{4}\,t_i^{\mu-2\alpha} \le |u''(\xi)| \le 2\sum_{j=0}^{2} \frac{|u(s_j)|}{(\frac{ht_i^{\alpha}}{2})^2} \le \frac{8cc_\mu}{h^2}\,t_i^{\mu-2\alpha},$$

where $c_\mu = 1$ if $\mu \geq 0$, and $c_\mu = (1-h)^\mu$ if $\mu < 0$. Therefore

$$M_i \leq \frac{32cc_\mu}{h^2}\,.$$

For any $i \in N$, which is a contradiction. In an analogous manner we obtain a contradiction when $l = 1$. The lemma is proved.

Remark 1. For $\alpha = \mu = 0$, Lemma 1 is proved in [1]. For second order equations see [2].

Lemma 2. *Let $\beta > 0$, $\alpha \geq 0$, let the conditions*

$$\limsup_{t\to+\infty} |p_k(t)| \exp(-\alpha k t^\beta) < +\infty \quad (k = 1, 2, 3)$$

be fulfilled and for some $\mu > 0$ let equations (1) *have a solution, satisfying the condition*

$$\limsup_{t\to+\infty} |u(t)| \exp(-\mu t^\beta) < +\infty.$$

Then

$$\limsup_{t\to+\infty} |u^{(j)}(t)| \exp(-(\mu + j\alpha)t^\beta) < +\infty \quad (j = 1, 2). \tag{8}$$

Proof. By transformation of the variable

$$u(t) = \exp(\mu t^\beta) v(s), \quad s = \int\limits_0^t \exp(\alpha\tau^\beta)\, d\tau, \tag{9}$$

equation (1) takes the form

$$v'''(s) + \widetilde{p}_1(s)v''(s) + \widetilde{p}_2(s)v'(s) + \widetilde{p}_3(s)v(s) = 0, \tag{10}$$

where

$$\begin{aligned}
\widetilde{p}_1(s) &= \Big(p_1(t) + \mu\beta t^{\beta-1} + (\mu+\alpha)\beta t^{\beta-1} + \beta(\mu+2\alpha)t^{\beta-1}\Big)\exp(-\alpha t^\beta),\\
\widetilde{p}_2(s) &= \Big[p_2(t) + p_1(t)\big(\mu\beta t^{\beta-1} + (\mu+\alpha)\beta t^{\beta-1}\big) + \mu\beta(\beta-1)t^{\beta-2} +\\
&\quad + \mu^2\beta^2 t^{2\beta-2} + \mu\beta(\beta-1)t^{\beta-2} + \mu(\mu+\alpha)\beta t^{2\beta-2} +\\
&\quad + (\mu+\alpha)\beta(\beta-1)t^{\beta-2} + (\mu+\alpha)^2\beta^2 t^{2\beta-2}\Big]\exp(-2\alpha t^\beta),\\
\widetilde{p}_3(s) &= \Big[p_3(t) + p_2(t)\mu\beta t^{\beta-1} +\\
&\quad + p_1(t)\big(\mu\beta(\beta-1)t^{\beta-2} + \mu^2\beta^2 t^{2\beta-2}\big) + \mu^3\beta^3 t^{3\beta-3}\Big]\exp(-3\alpha t^\beta).
\end{aligned}$$

It is obvious that for equation (10) the conditions of Lemma 1 are fulfilled if it is assumed that $\mu = 0$ and $\alpha = 0$. Therefore

$$\limsup_{t\to+\infty} |v^{(j)}(s)| < +\infty \quad (j = 1, 2).$$

This, by virtue of (9), implies inequality (8). The Lemma is proved.

Theorem 1. *If the inequalities*

$$p_2(t) \le 0, \quad p_3(t) \ge 0 \ \ for \ \ t \in R_+, \tag{11}$$

$$\int\limits_0^{+\infty} [p_1(t)]_+ \, dt < +\infty \tag{12}$$

are fulfilled, then there exists a solution of equation (1) *such that*

$$\limsup_{t\to+\infty} t^{-\frac{3}{2}+j} |u^{(j)}(t)| > 0 \ \ (j = 1, 2). \tag{13}$$

If, besides, condition (2) *is fulfilled for some* $\alpha \le 1$*, then equation* (1) *has a solution which, in addition to* (13)*, also satisfies the condition*

$$\limsup_{t\to+\infty} t^{-1-\frac{\alpha}{2}} |u(t)| > 0. \tag{14}$$

Proof. Let u_1 and u_2 be solutions of equation (1) which satisfy the initial conditions

$$u_1(0) = 0, \quad u_1'(0) = 1, \quad u_1(0) = 0,$$
$$u_2(0) = 0, \quad u_2'(0) = 0, \quad u_2''(0) = 1.$$

Let us introduce the notation

$$v_{01}(t) = u_1(t)u_2'(t) - u_1'(t)u_2(t),$$

$$v_{02}(t) = \exp\bigg(\int\limits_0^t [p_1(s)]_+ \, ds\bigg)\big(u_1(t)u_2''(t) - u_1''(t)u_2(t)\big),$$

$$v_{12}(t) = \exp\bigg(\int\limits_0^t [p_1(s)]_+ \, ds\bigg)\big(u_1'(t)u_2''(t) - u_1''(t)u_2'(t)\big).$$

The vector-function $x = colon(v_{01}, v_{02}, v_{12})$ is a solution of the problem

$$x' = A(t)x, \quad x(0) = colon(0, 0, 1),$$

where

$$A(t) = \begin{pmatrix} 0 & \exp\bigg(-\int\limits_0^t [p_1(s)]_+ \, ds\bigg) & 0 \\ -p_2(t)\exp\bigg(\int\limits_0^t [p_1(s)]_+ \, ds\bigg) & [p_1(t)]_- & 1 \\ p_3(t)\exp\bigg(\int\limits_0^t [p_1(s)]_+ \, ds\bigg) & 0 & [p_1(t)]_- \end{pmatrix}.$$

Let

$$y(t) = colon\Big(\int_0^t s\exp\Big(-\int_0^s [p_1(\tau)]_+\,d\tau\Big)\,ds, t, 1\Big).$$

Then y satisfies the system

$$y' = B(t)y,$$

where

$$B(t) = \begin{pmatrix} 0 & \exp\Big(-\int_0^t [p_1(s)]_+\,ds\Big) & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $x(0) \geq y(0) \geq 0$ and

$$A(t) \geq B(t) \geq 0 \ \text{ for } \ t \geq 0$$

it is easy to show that

$$x(t) \geq y(t) \ \text{ for } \ t \geq 0.$$

Therefore

$$v_{01}(t) \geq \int_0^t s\exp\Big(-\int_0^s [p_1(\tau)]_+\,d\tau\Big)\,ds \ \text{ for } \ t \geq 0.$$

With (12) taken into account, we obtain

$$\limsup_{t\to+\infty} \frac{v_{01}(t)}{t^2} > 0. \tag{15}$$

Let us show that u_1 or u_2 satisfies condition (13). Indeed, assuming the contrary, we have

$$\lim_{t\to+\infty} t^{-\frac{1}{2}} u_i'(t) = \lim_{t\to+\infty} t^{-\frac{3}{2}} u_i(t) = 0 \ \ (i = 1, 2),$$

which contradicts condition (15).

Now assume that conditions (2) are fulfilled, then

$$\lim_{t\to+\infty} t^{-1-\frac{\alpha}{2}} u_i(t) = 0 \ \ (i = 1, 2).$$

In that case, by virtue of Lemma 1

$$\limsup_{t\to+\infty} t^{-1+\frac{\alpha}{2}} u_i'(t) < +\infty \ \ (i = 1, 2)$$

and therefore

$$\lim_{t\to+\infty} t^{-2} v_{01}(t) = 0,$$

which contradicts inequality (15). The theorem is proved.

Corollaries 1.2.1, 1.3.1 (see [3], pp. 453, 455]) and Theorem 1 immediately give rise to the following propositions.

Corollary 1.1. Let $\alpha < 1$, conditions (11),(12) and

$$\lim_{t\to+\infty} t^{k\alpha} p_k(t) = 0 \ \ (k = 1, 2),$$
$$0 < \liminf_{t\to+\infty} t^{3\alpha} p_3(t) \le \limsup_{t\to+\infty} t^{3\alpha} p_3(t) < +\infty$$

be fulfilled. Then equation (1) has an oscillatory solution which satisfies conditions (13) and (14).

Theorem 2. *Let* (11),(12) *and let one of the following two conditions*

$$\lim_{t\to+\infty} t^3 p_3(t) = +\infty \tag{16}$$

or

$$\lim_{t\to+\infty} t^2 p_2(t) = +\infty \tag{17}$$

be fulfilled. Then equation (1) *has a solution such that*

$$\limsup_{t\to+\infty} t^{-\mu} |u^{(j)}(t)| = +\infty \tag{18}$$

for any $\mu > 0$ *and* $j \in \{1, 2\}$*. If, besides, conditions* (2) *hold for some* $\alpha < 1$*, then there exists a solution of equation* (1) *which satisfies condition* (18) *for any* $\mu > 0$ *and* $j \in \{0, 1, 2\}$*.*

Proof. It is analogous to the proof of Theorem 1, now for $t \ge t_0 > 0$ we put

$$B(t) = \begin{pmatrix} 0 & \exp\Big(-\int\limits_0^t [p_1(s)]_+ ds\Big) & 0 \\ 0 & 0 & 1 \\ \dfrac{\nu(\nu-1)t^{\nu-2}}{\int\limits_0^t s^\nu \exp\big(-\int\limits_0^s [p_1(\tau)]_+ \, d\tau\big)\, ds} & 0 & 0 \end{pmatrix},$$

$$y(t) = colon\Big(\int\limits_0^t s^\nu \exp\Big(-\int\limits_0^s [p_1(\tau)]_+ \, d\tau\Big)\, ds, t^\nu, \nu t^{\nu-1}\Big)$$

if conditions (16) are fulfilled, and

$$B(t) = \begin{pmatrix} 0 & \exp\Big(-\int\limits_0^t [p_1(s)]_+ \, ds\Big) & 0 \\ \dfrac{\nu t^{\nu 1}}{\int\limits_0^t s^\nu \exp\big(-\int\limits_0^s [p_1(\tau)]_+ \, d\tau\big)\, ds} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$y(t) = colon\Big(\int\limits_0^t s^\nu \exp\Big(-\int\limits_0^s [p_1(\tau)]_+ \, d\tau\Big)\, ds, t^\nu, 1\Big)$$

if (17) is fulfilled.

Remark 2. In Theorems 1 and 2, the requirement that $p_2(t) \le 0$ for $t \ge 0$ is an essential one.

Indeed, let us consider the differential equation

$$u''' - u'' + \frac{1}{4}\,u' + \frac{9}{4}\,u = 0, \tag{19}$$

which has a fundamental system of solutions

$$e^{-t}, \quad e^{-t}\sin\frac{\sqrt{5}}{2}\,t, \quad e^{-t}\cos\frac{\sqrt{5}}{2}\,t.$$

Thus equation (19) has no unbounded solution though all the conditions of Theorems 1 and 2 are fulfilled except the condition that the function p_2 is non-positive.

According to Theorem 3.2 [5], Theorem 2 immediately implies

Corollary 2.1. Let $\alpha < 1$, let conditions (11), (12), (16) and

$$\limsup_{t\to+\infty} t^k |p_k(t)| < +\infty \ \ (k = 1, 2), \quad \limsup_{t\to+\infty} t^{3\alpha} p_3(t) < +\infty$$

be fulfilled. Then equation (1) has an oscillatory solution, satisfying conditions (18) for any $\mu > 0$ and $j \in \{0, 1, 2\}$.

Theorem 3. *Let* $\sigma > 0$,

$$\limsup_{t\to+\infty} t^{-\sigma} \int_0^t [p_1(s)]_+\,ds < +\infty, \tag{20}$$

let inequality (11) *and one of the following two conditions*

$$\lim_{t\to+\infty} t^{3-3\sigma} p_3(t) = +\infty \tag{21}$$

or

$$\lim_{t\to+\infty} t^{2-2\sigma} |p_2(t)| = +\infty \tag{22}$$

be fulfilled. Then (1) *has a solution such that*

$$\limsup_{t\to+\infty} |u^{(j)}(t)| \exp(-\mu t^\sigma) = +\infty \tag{23}$$

for any $\mu > 0$ *and* $j \in \{1, 2\}$. *If, besides, for some* $\alpha \ge 0$

$$\limsup_{t\to+\infty} |p_k(t)| \exp(-\alpha k t^\sigma) < +\infty \ \ (k = 1, 2, 3), \tag{24}$$

Then there exists a solution of equation (1) *which satisfies condition* (23) *for any* $\mu > 0$ *and* $j \in \{0, 1, 2\}$.

Proof. We begin by assuming that condition (21) is fulfilled. Let $\mu > 0$ and u_1, u_2, v_{01}, v_{02}, v_{12}, x, A be defined as they were in proving Theorem 1, and let ν be chosen so that

$$\int_0^t [p_1(s)]_+\,ds \le (\nu - 2\mu)t^\sigma \ \text{ for } \ t \ge t_0. \tag{25}$$

Put

$$y(t) =$$

$$= colon\Big(\int_0^t \exp\Big(-\int_0^s [p_1(\tau)]_+\,d\tau\Big)\exp(\nu s^\sigma)\,ds, \exp(\nu t^\sigma), \nu\sigma t^{\sigma-1}\exp(\nu t^\sigma)\Big).$$

Then y on the interval $]0,+\infty[$ satisfies the system

$$y' = B(t)y,$$

where

$$B(t) = \begin{pmatrix} 0 & \exp\Big(-\int_0^t [p_1(s)]_+\,ds\Big) & 0 \\ 0 & 0 & 1 \\ b_{\sigma\nu}(t) & 0 & 0 \end{pmatrix},$$

$$b_{\sigma\nu}(t) = \frac{\big(\nu(\sigma-1)\sigma t^{\sigma-2} + \nu^2\sigma^2 t^{2\sigma-2}\big)\exp(\nu t^\sigma)}{\int_0^t \exp\big(-\int_0^s [p_1(\tau)]_+\,d\tau\big)\exp(\nu s^\sigma)\,ds}.$$

By (21) it is easy to verify that

$$\limsup_{t\to+\infty} \frac{p_3(t)\exp\big(\int_0^t [p_1(s)]_+\,ds\big)}{b_{\sigma\nu}(t)} = +\infty.$$

If $\varepsilon > 0$ is such that

$$x(t_0) \geq \varepsilon y(t_0) \geq 0,$$
$$A(t) \geq B(t) \geq 0 \ \text{ for } \ t \geq t_0,$$

then it can be easily shown that

$$x(t) \geq \varepsilon y(t) \ \text{ for } \ t \geq t_0.$$

Therefore

$$v_{01}(t) \geq \varepsilon \int_0^t \exp\Big(-\int_0^s [p_1(\tau)]_+\,d\tau\Big)\exp(\nu s^\sigma)\,ds \ \text{ for } \ t \geq t_0.$$

Hence by virtue of (25) we obtain

$$\limsup_{t\to+\infty} \frac{v_{01}(t)}{t^{1-\sigma}\exp(2\mu t^\sigma)} = +\infty. \tag{26}$$

Let us show that u_1 or u_2 satisfies condition (23). Indeed, if we assume the contrary, we have

$$\limsup_{t\to+\infty} |u_i'(t)| \exp(-\mu t^\sigma) < +\infty \quad (i = 1, 2),$$
$$\limsup_{t\to+\infty} |u_i(t)| t^{\sigma-1} \exp(-\mu t^\sigma) < +\infty \quad (i = 1, 2).$$

Then

$$\limsup_{t\to+\infty} v_{01}(t) t^{\sigma-1} \exp(-2\mu t^\sigma) < +\infty,$$

which contradicts (26). Thus u_1 or u_2 satisfies condition (23).

If, besides, (24) holds and

$$\limsup_{t\to+\infty} |u_i(t)| \exp(-\mu t^\sigma) < +\infty \quad (i = 1, 2),$$

then by virtue of Lemma 2 we obtain

$$\limsup_{t\to+\infty} |u_i'(t)| \exp(-(\mu+\alpha) t^\sigma) < +\infty \quad (i = 1, 2)$$

and

$$\limsup_{t\to+\infty} v_{01}(t) \exp(-(2\mu+\alpha) t^\sigma) < +\infty.$$

But, as above, this is a contradiction.

Now assume that condition (22) is fulfilled. Then the proof is carried out as above, only in this case

$$B(t) = \begin{pmatrix} 0 & \exp\left(-\int\limits_0^t [p_1(s)]_+ \, ds\right) & 0 \\ \dfrac{\nu\sigma t^{\sigma-1}\exp(\nu t^\sigma)}{\int\limits_0^t \exp\big(-\int\limits_0^s [p_1(\tau)]_+ \, d\tau\big)\exp(\nu s^\sigma)\, ds} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$y(t) = colon\left(\int\limits_0^t \exp\left(-\int\limits_0^s [p_1(\tau)]_+ \, d\tau\right) \exp(\nu s^\sigma) ds, \exp(\nu t^\sigma), 1\right).$$

The theorem is proved.

According to Theorem 3.2 [5], Theorem 3 immediately implies

Corollary 3.1. Let conditions (11), (20) be fulfilled and

$$\limsup_{t\to+\infty} |p_k(t)| < +\infty \quad (k = 1, 2), \qquad \lim_{t\to+\infty} p_3(t) = +\infty,$$
$$\limsup_{t\to+\infty} p_3(t) \exp(-3\alpha t^\sigma) < +\infty.$$

Then equation (1) has an oscillatory solution, satisfying conditions (23) for any $j \in \{0, 1, 2\}$.

In conclusion, we present a theorem on an asymptotic oscillatory solution of equation (1) when p_3 is a non-negative function.

Theorem 4. *If equation* (1) *is oscillatory,*

$$p_1(t) \ge 0, \quad p_2(t) \le 0, \quad p_3(t) \ge 0 \ \textit{for} \ t \ge 0 \tag{27}$$

and

$$\int_0^{+\infty} p_1(t)\,dt < +\infty,$$

then equation (1) *has a non-oscillatory solution and any of such solutions satisfies the condition*

$$u(t)u'(t) \le 0 \ \textit{for} \ t \ge 0, \quad \lim_{t\to+\infty} u(t) = 0. \tag{28}$$

To prove this theorem we need lemmas on the asymptotic properties of solutions of the differential equation

$$\Big(\frac{1}{a_2(t)}\Big(\frac{x'}{a_1(t)}\Big)'\Big)' + p(t)x = 0, \tag{29}$$

where $a_i(t) : R_+ \to]0,+\infty[$ $(i = 1,2)$, $p : R_+ \to R_+$ are continuous functions.

Lemma 3. *Let*

$$\int_0^{+\infty} a_2(t)\,dt = +\infty, \quad \int_0^{+\infty} a_1(t) \int_0^t a_2(s)\,ds\,dt = +\infty \tag{30}$$

and equation (1) *have the solution* x *which for some* $t_0 \ge 0$ *satisfies the conditions*

$$x(t) > 0, \quad x'(t) > 0, \quad \Big(\frac{1}{a_1(t)}x'(t)\Big)' > 0 \ \textit{for} \ t \ge t_0.$$

Then equation (29) *is non-oscillatory.*

For the proof of this lemma see ([6], Lemma 4.2).

Lemma 4. *If* p *is not identically zero in the neighborhood of* $+\infty$, *conditions* (30) *are fulfilled and* x *is a solution of equation* (29) *that satisfies the inequality*

$$x(t) > 0 \ \textit{for} \ t \ge t_0. \tag{31}$$

Then there exists $t_1 \ge t_0$ *such that either*

$$x'(t) > 0, \quad \Big(\frac{1}{a_1(t)}\,x'(t)\Big)' > 0 \ \textit{for} \ t \ge t_1$$

or

$$x'(t) < 0, \quad \Big(\frac{1}{a_1(t)}\,x'(t)\Big)' > 0 \ \textit{for} \ t \ge t_0.$$

Proof. To prove the lemma it suffices to show that

$$\frac{1}{a_2(t)}\Big(\frac{1}{a_1(t)}\,x'(t)\Big)' > 0 \ \text{ for } \ t \geq t_0. \tag{32}$$

Since $p(t) \geq 0$, the function

$$\frac{1}{a_2}\Big(\frac{1}{a_1}\,x'\Big)'$$

does not increase. If (32) does not hold, then since p is not identically zero in the neighborhood of ∞, there are $t_1 \geq t_0$ and $c_0 < 0$ such that

$$\frac{1}{a_2(t)}\Big(\frac{1}{a_1(t)}\,x'(t)\Big)' \leq c_0 \ \text{ for } \ t \geq t_1.$$

This inequality readily implies that

$$x(t) \leq c_0 \int_{t_1}^{t} a_1(s_1) \int_{t_1}^{s_1} a_2(s_2)\, ds_2\, ds_1 + \frac{x'(t_1)}{a_1(t_1)} \int_{t_1}^{t} a_1(s)\, ds + x(t_1) \ \text{ for } \ t \geq t_1.$$

If in the latter inequality we pass to the limit as $t \to +\infty$, then, taking (30) into account, we have

$$\lim_{t\to+\infty} x(t) = -\infty.$$

The obtained contradiction proves (32). The lemma is proved.

Lemma 5. *Let condition* (30) *be fulfilled. Then for the existence of a solution x of equation* (29) *that satisfies the condition*

$$\lim_{t\to+\infty} x(t) = 1, \tag{33}$$

it is necessary and sufficient that

$$\int_0^{+\infty}\int_0^{s_3} a_2(s_2) \int_0^{s_2} a_1(s_1)\, ds_1\, ds_2 p(s_3)\, ds_3 < +\infty. \tag{34}$$

Proof. *Sufficiency.* Choose such a large t_0 that

$$\int_{t_0}^{+\infty}\int_{t_0}^{s_3} a_2(s_2) \int_{t_0}^{s_2} a_1(s_1)\, ds_1\, ds_2 p(s_3)\, ds_3 = \Theta < 1.$$

Let

$$S = \Big\{x \in C([t_0, +\infty[) :\ 0 \leq x(t) \leq 2 \text{ for } t \geq t_0\Big\}.$$

Consider the integral operator $F : S \to S$ defined by the equality

$$F(x)(t) = 1 + \int_t^{+\infty}\int_t^{s_3} a_2(s_2) \int_t^{s_2} a_1(s_1)\, ds_1\, ds_2 p(s_3) x(s_3)\, ds_3.$$

If $u, v \in S$, then

$$\begin{aligned}&\big|F(u)(t) - F(v)(t)\big|\\ \le &\bigg|\int\limits_t^{+\infty}\int\limits_t^{s_3} a_2(s_2)\int\limits_t^{s_2} a_1(s_1)\,ds_1\,ds_2 p(s_3)\big(u(s_3)-v(s_3)\big)\,ds_3\bigg|\\ &\le \|u-v\|\cdot\Theta \ \text{ for } \ t \ge t_0.\end{aligned}$$

This means that F is a contracting operator and by virtue of the well-known Banach theorem, F has a fixed point, i.e. there exists $x \in S$ such that

$$x(t) = 1 + \int\limits_t^{+\infty}\int\limits_t^{s_3} a_2(s_2)\int\limits_t^{s_2} a_1(s_1)\,ds_1\,ds_2 p(s_3)\,ds_3 \ \text{ for } \ t \ge t_0.$$

It is easy to verify that x is a solution of equation (29) that satisfies (33).

Necessity. Assume that x is a solution of equation (29) that satisfies condition (33). Then by virtue of Lemma 4 there exists $t_0 > 0$ such that

$$x(t) > 0, \quad x'(t) < 0, \quad \Big(\frac{1}{a_1(t)}\,x'(t)\Big)' > 0 \ \text{ for } \ t \ge t_0.$$

The equality

$$\begin{aligned}&\int\limits_{t_0}^{t}\int\limits_{t_0}^{s} a_2(s_2)\int\limits_{t_0}^{s_2} a_1(s_1)\,ds_1\,ds_2 p(s)x(s)\,ds\\ &\qquad = -\int\limits_{t_0}^{t} a_2(s_2)\int\limits_{t_0}^{s_2} a_1(s_1)\,ds_1\,ds_2\,\frac{1}{a_2(t)}\Big(\frac{x'(t)}{a_1(t)}\Big)'\\ &\qquad\qquad + \int\limits_{t_0}^{t} a_1(s_1)\,ds_1\,\frac{x'(t)}{a_1(t)} - x(t) + x(t_0) \ \text{ for } \ t \ge t_0\end{aligned}$$

implies (34). The lemma is proved.

Lemma 6. *Let condition* (30) *be fulfilled. Then for the existence of a solution x of equation* (29) *that satisfies the condition*

$$\lim_{t\to+\infty}\frac{x(t)}{\int\limits_0^t a_1(s)\int\limits_0^s a(\tau)\,d\tau\,ds} = 1. \tag{35}$$

It is necessary and sufficient that

$$\int\limits_0^{+\infty} p(s_3)\int\limits_0^{s_3} a_1(s_1)\int\limits_0^{s_1} a_2(s_2)\,ds_2\,ds_1\,ds_3 < +\infty. \tag{36}$$

Proof. The *sufficiency* is proved as in Lemma 5, but in this case the set S and the operator $F : S \to S$are defined as follows

$$S = \Big\{ u \in C([t_0, +\infty[) : \ 0 \le u(t) \le \int_0^t a_1(s) \int_0^s a_2(\tau)\, d\tau\, ds \text{ for } t \ge t_0 \Big\},$$

$$F(u)(t) = \int_{t_0}^t a_1(s_1) \int_{t_0}^{s_1} a_2(s_2)\, ds_2\, ds_1 + \int_{t_0}^t a_1(s_1) \int_{t_0}^{s_1} a_2(s_2) \int_{s_2}^{+\infty} p(s_3) u(s_3)\, ds_3\, ds_2\, ds_1.$$

Necessity. If x is a solution of equation (29) that satisfies condition (35), then, taking into account Lemma 4, we obtain

$$x(t) > 0, \quad x'(t) > 0, \quad \Big(\frac{1}{a_1(t)}\, x'(t) \Big)' > 0 \ \text{ for } \ t \ge t_0.$$

Then by virtue of (35) from the equality

$$\int_{t_0}^t p(s) x(s)\, ds = -\frac{1}{a_2(t)} \Big(\frac{x'(t)}{a_1(t)} \Big)' + \frac{1}{a_2(t)} \Big(\frac{x'(t)}{a_1(t)} \Big)' \Big|_{t = t_0}$$

we have (36). The lemma is proved.

Lemma 7. *Let equation* (29) *be oscillatory and let condition* (30) *be fulfilled. In addition to this, assume that there is a number* $c > 0$ *such that the inequality*

$$\frac{a_1(s_1)}{a_2(s_1)} \ge \frac{a_1(s_2)}{a_2(s_2)} \cdot c$$

holds for for any $s_1 > 0$ *and* $s_2 > 0$*, where* $s_1 \le s_2$*. Then equation* (29) *has a non-oscillatory solution and any such solution tends to zero at infinity.*

Proof. The existence of a non-oscillatory solution follows from Theorem 14.2.1 in [7]. Since equation (29) is oscillatory, by virtue of Lemmas 3, 4, 6

$$\int_0^{+\infty} p(s_3) \int_0^{s_3} a_1(s_1) \int_0^{s_1} a_2(s_2)\, ds_2\, ds_1\, ds_3 = +\infty.$$

Then, since

$$\int_0^{+\infty}\int_0^{s_3} a_2(s_2)\int_0^{s_2} a_1(s_1)\,ds_1\,ds_2 p(s_3)\,ds_3$$
$$= \int_0^{+\infty}\int_0^{s_3} a_2(s_2)\int_0^{s_2} \frac{a_1(s_1)}{a_2(s_1)}\,a_2(s_1)\,ds_1\,ds_2 p(s_3)\,ds_3$$
$$\geq c\int_0^{+\infty}\int_0^{s_3} a_1(s_2)\int_0^{s_2} a_2(s_1)\,ds_1\,ds_2 p(s_3)\,ds_3,$$

we have

$$\int_0^{+\infty} p(s_3)\int_0^{s_3} a_2(s_2)\int_0^{s_2} a_1(s_1)\,ds_1\,ds_2\,ds_3 = +\infty.$$

Therefore, if x is a non-oscillatory solution of equation (29), by virtue of Lemmas 3, 4, 5

$$\lim_{t\to+\infty} x(t) = 0.$$

The lemma is proved.

Proof. [Proof of Theorem 4] Equation (1) on the interval $[0,+\infty[$ can be written in the form (29), where

$$p(t) = p_3(t)v(t)\exp\bigg(\int_0^t p_1(s)\,ds\bigg),$$

a_1, a_2 are defined by the equalities

$$a_1(t) = v(t), \quad a_2(t) = v^{-2}(t)\exp\bigg(-\int_0^t p_1(\tau)\,d\tau\bigg),$$

and v is a solution of the equation

$$(g(t)v')' + q(t) = 0,$$

where

$$g(t) = \exp\bigg(\int_0^t p_1(\tau)\,d\tau\bigg), \quad q(t) = g(t)p_2(t),$$

which satisfies the condition

$$v(t) > 0, \quad v'(t) \leq 0 \ \text{ for } \ t \geq 0.$$

Then, as is known (see [7, pp. 419–422]), condition (30) is fulfilled.

Moreover,

$$\frac{a_1(s_1)}{a_2(s_1)} = \frac{a_1(s_2)}{a_2(s_2)} \cdot \frac{v_1^3(s_1)}{v_1^3(s_2)} \exp\Big(-\int_{s_1}^{s_2} p_1(\tau)\,d\tau\Big) \ge \frac{a_1(s_2)}{a_2(s_2)} \cdot c \ \text{ for } \ s_2 \ge s_1 \ge 0,$$

where

$$c = \exp\Big(-\int_0^{+\infty} p_1(\tau)\,d\tau\Big).$$

Thus all the conditions of Lemma 7 are fulfilled. This lemma immediately implies the validity of the theorem.

Remark 3. In Theorem 4 the condition $p_2(t) \le 0$ for $t \ge 0$ is an essential one.

Indeed, let us consider the equation

$$u''' + \frac{1}{4t^2}\,u' + \frac{c}{t^3 \ln^{3/2} t}u = 0 \ \ (t \ge a > 1), \tag{37}$$

where $c > 0$. By Theorem 5 [8] this equation is oscillatory. Equation (37) can be written in the form (29), where

$$a_1(t) = t^{\frac{1}{2}}, \quad a_2(t) = \frac{1}{t}, \quad p(t) = \frac{c}{t^{5/2} \ln^{3/2} t}.$$

Since

$$\int_a^{+\infty}\int_a^{s_3} a_2(s_2)\int_a^{s_2} a_1(s_1)\,ds_1\,ds_2 p(s_3)\,ds_3 < +\infty.$$

By virtue of Lemma 5, equation (37) has a solution, satisfying condition (33).

Corollaries 1.1, 2.1 and Theorem 4 immediately give rise to the following propositions.

Corollary 4.1. Let $\alpha < 1$, conditions (27) be fulfilled and

$$\int_0^{+\infty} p_1(t)\,dt < +\infty, \quad \lim_{t\to+\infty} t^{k\alpha} p_k(t) = 0 \ \ (k = 1, 2),$$

$$0 < \liminf_{t\to+\infty} t^{3\alpha} p_3(t) \le \limsup_{t\to+\infty} t^{3\alpha} p_3(t) < +\infty.$$

Then equation (1) has both non-oscillatory solutions, satisfying condition (28) and oscillatory solutions, satisfying conditions (13), (14).

Corollary 4.2. Let conditions (27) be fulfilled and

$$\int_0^{+\infty} p_1(t)\,dt < +\infty, \quad \lim_{t\to+\infty} t^{k} p_k(t) = 0 \ \ (k = 1, 2),$$

$$\frac{2\sqrt{3}}{9} < \liminf_{t\to+\infty} t^{3} p_3(t) \le \limsup_{t\to+\infty} t^{3} p_3(t) < +\infty.$$

Then equation (1) has both oscillatory solutions satisfying both condition (28) and conditions (13), (14).

Remark 4. From the results of [9] (see also [10], [11]) it follows that under the conditions of Theorem 4, the solution of equation (1), satisfying condition (28), is unique to within a constant multiplier.

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