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EFFECTIVE SOLUTION OF ONE BOUNDARY VALUE PROBLEM OF THE LINEAR THEORY OF THERMOELASTICITY WITH MICROTEMPERATURES FOR A SPHERICAL RING

Bitsadze L.

Abstract. In this paper the expansion of regular solution for the equations of the theory of thermoelasticity with microtemperatures is obtained, that we use for explicitly solving one basic boundary value problem (BVP) of the linear equilibrium theory of thermoelasticity with microtemperatures for the spherical ring. The obtained solutions are represented as absolutely and uniformly convergent series.

Keywords and phrases: Thermoelasticity with microtemperatures, absolutely and uniformly convergent series, spherical harmonic.

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Introduction

The linear theory of thermoelasticity for materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures was constructed by Iesan and Quintanilla [1]. The fundamental solutions of the equations of the three-dimensional (3D) theory of thermoelasticity with microtemperatures were constructed by Svanadze [2]. The representations of the Galerkin type and general solutions of the system in this theory were obtained by Scalia, Svanadze and Tracinà [3]. The 3D linear theory of thermoelasticity for microstretch elastic materials with microtemperatures was constructed by Iesan [4] where the uniqueness and existence theorems in the dynamical case for isotropic materials are proved.

The purpose of this paper is to solve explicitly one basic boundary value problem (BVP) of the linear equilibrium theory of thermoelasticity with microtemperatures for the spherical ring. The obtained solutions are represented as absolutely and uniformly convergent series.

Basic equations

Let D be a bounded (respectively, an unbounded) domain of the Euclidean 3D space E_3, bounded by the surface S. Let $\mathbf{x} = (x_1, x_2, x_3) \in E_3$, $\rho = |\mathbf{x}|$, $\partial\mathbf{x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$. Assume that the domain D is filled with isotropic elastic materials with the thermoelastic properties possessing microtemperatures.

The basic homogeneous (i.e., body forces are neglected) system of equations of the linear equilibrium theory of thermoelasticity with microtemperatures has the form [1]

$$\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - \beta grad\theta = 0, \tag{1}$$

$$k_6\Delta\mathbf{w} + (k_4+k_5) graddiv\mathbf{w} - k_3 grad\theta - k_2\mathbf{w} = 0, \tag{2}$$

$$k\Delta\theta + k_1 div\mathbf{w} = 0, \tag{3}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$ is the displacement vector, $\mathbf{w} = (w_1, w_2, w_3)^T$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0 $(T_0 > 0)$ by the natural state (i.e. by the state of the absence of loads), λ, μ, β, k, k_j, $j = 1, ..., 6$, are constitutive coefficients, Δ is the 3D Laplace operator. The superscript "T" denotes transposition.

Definition 1. A vector-function $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \theta)$ defined in the domain D is called regular if it has integrable continuous second order derivatives in D , and $\mathbf{U}$ itself and its first order derivatives are continuously extendible at every point of the boundary of D, that is $\mathbf{U} \in C^2(D) \cap C^1(\overline{D})$.

Note that BVPs for the system (2),(3), that contain only $\mathbf{w}$ and θ, can be investigated separately. Then supposing θ, as known, we can study BVPs for the system (1) with respect to $\mathbf{u}$. Combining the results obtained we arrive at explicit solution for BVPs for the system (1)-(3). First we assume that $\theta(\mathbf{x})$ is known, when $\mathbf{x} \in D$, then for $\mathbf{u}$ we get the following nonhomogeneous equation

$$\mu\Delta\mathbf{u} + (\lambda + \mu) graddiv\mathbf{u} = \beta grad\theta. \tag{4}$$

It is known that the volume potential $\mathbf{u}_0$ [6]

$$\mathbf{u}_0 = -\frac{\beta}{\pi} \int\limits_D \mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y}) grad\theta ds, \tag{5}$$

where

$$\mathbf{\Gamma} = \|\Gamma_{kj}\|_{3x3},$$

$$\Gamma_{kj} = \frac{\lambda + 3\mu}{2a\mu}\frac{\delta_{kj}}{r} + \frac{\lambda + \mu}{2a\mu}\frac{x_k x_j}{r^3}, \quad k, j = 1, 2, 3.$$

is a particular solution of (4). In (5) $grad\theta$ is a continuous vector in D along with its first order derivatives.

Thus, the general solution of the equation (4) is representable in the form $\mathbf{u} = \mathbf{V} + \mathbf{u}_0$ where

$$\mu\Delta\mathbf{V} + (\lambda + \mu) graddiv\mathbf{V} = 0. \tag{6}$$

The last equation is the equation of an isotropic elastic body. So we have reduced the solution of basic BVPs under consideration to the solution of the basic BVPs for the equation of an isotropic elastic body.

The solution of the BVPs for the equation (6) is given in [6]. So it remains to solve BVPs for the system (2),(3).

Expansion of regular solutions

In this section the general solution for the equations (2),(3) is obtained that gives possibility to solve the BVP for the spherical ring.

Theorem 1. *The regular solution $\boldsymbol{W} = (\boldsymbol{w}, \theta)$ of equations (2),(3) admits in the domain of regularity a representation*

$$\boldsymbol{W}(\boldsymbol{x}) = (\overset{1}{\mathbf{w}} + \overset{2}{\mathbf{w}}, \theta), \tag{7}$$

where

$$\Delta(\Delta-s_1^2)\overset{1}{\mathbf{w}}=0,\quad rot\overset{1}{\mathbf{w}}=0,\quad (\Delta-s_1^2)div\overset{1}{\mathbf{w}}=0,\quad (\Delta-s_2^2)\overset{2}{\mathbf{w}}=0,$$
$$div\overset{2}{\mathbf{w}}=0,\quad \Delta(\Delta-s_1^2)\theta=0,\quad s_1^2=\frac{kk_2-k_1k_3}{kk_7}>0,\quad s_2^2=\frac{k_2}{k_6}>0. \tag{8}$$

Proof. Let $\mathbf{W}$ be certain solution of the equation (2),(3). Let us prove that $\mathbf{W}$ can be represented in the form (7) and it satisfies the conditions (8). Using the identity

$$\Delta\mathbf{w}=graddiv\mathbf{w}-rotrot\mathbf{w}$$

rewrite equation (2) as follows

$$\mathbf{w}=\frac{k_7}{k_2}graddiv\mathbf{w}-\frac{k_6}{k_2}rotrot\mathbf{w}-\frac{k_3}{k_2}grad\theta.$$

Let

$$\overset{1}{\mathbf{w}}=\frac{k_7}{k_2}graddiv\mathbf{w}-\frac{k_3}{k_2}grad\theta, \tag{9}$$

$$\overset{2}{\mathbf{w}}=-\frac{k_6}{k_2}rotrot\mathbf{w}. \tag{10}$$

Clearly, from (9),(10) we obtain

$$rot\overset{1}{\mathbf{w}}=0,\quad div\overset{2}{\mathbf{w}}=0,\quad (\Delta-s_2^2)\overset{2}{\mathbf{w}}=0. \tag{11}$$

(2),(3) yield

$$(k_7\Delta-k_2)div\mathbf{w}-k_3\Delta\theta=0. \tag{12}$$

Substitution of the value $div\mathbf{w}=-\frac{k}{k_1}\Delta\theta$ from (3) in (12) results in

$$\Delta(\Delta-s_1^2)\theta=0. \tag{13}$$

From (9) and (10) we have

$$\Delta(\Delta-s_1^2)\overset{1}{\mathbf{w}}=0\quad (\Delta-s_1^2)div\overset{1}{\mathbf{w}}=0. \tag{14}$$

Formulas (11),(13),(14) prove the theorem.

Theorem 2. *In the domain of regularity the regular solution of equations (2),(3) can be represented in the form*

$$\boldsymbol{W}=\overset{1}{\mathbf{V}}+\overset{2}{\mathbf{V}}+\overset{3}{\mathbf{V}}, \tag{15}$$

where

$$\overset{1}{\mathbf{V}}=(\boldsymbol{v}^{(1)},\varphi_1),\quad \overset{2}{\mathbf{V}}=(\boldsymbol{v}^{(2)},\varphi_2),\quad \overset{3}{\mathbf{V}}=(\boldsymbol{v}^{(3)},0) \tag{16}$$

and

$$\Delta\mathbf{v}^{(1)}=0,\quad (\Delta-s_1^2)\mathbf{v}^{(2)}=0,\quad (\Delta-s_2^2)\mathbf{v}^{(3)}=0,\quad rot\mathbf{v}^{(1)}=0,$$

$$rot\mathbf{v}^{(2)} = 0, \quad div\mathbf{v}^{(3)} = 0, \qquad \Delta\varphi_1 = 0, \qquad (\Delta - s_1^2)\varphi_2 = 0.$$

Proof. Let

$$\overset{1}{\mathbf{v}} = -\frac{(\Delta - s_1^2)\overset{1}{\mathbf{w}}}{s_1^2}, \qquad \overset{2}{\mathbf{v}} = \frac{\Delta\overset{1}{\mathbf{w}}}{s_1^2}, \qquad \varphi_1 = -\frac{(\Delta - s_1^2)\theta}{s_1^2}, \qquad \varphi_2 = \frac{\Delta\theta}{s_1^2}, \tag{17}$$

then, by virtue of (14), it follows

$$\overset{1}{\mathbf{v}} + \overset{2}{\mathbf{v}} = \overset{1}{\mathbf{w}}, \qquad \Delta\overset{1}{\mathbf{v}} = 0, \qquad (\Delta - s_1^2)\overset{2}{\mathbf{v}} = 0.$$

θ is the solution of a scalar equation of the same type that it satisfied by the vector $\mathbf{w}^{(1)}$; therefore, by analogy we will have $\theta = \varphi_1 + \varphi_2$, where

$$\Delta\varphi_1 = 0, \quad (\Delta - s_1^2)\varphi_2.$$

Now, it is clear that if we take $\mathbf{v}^{(3)} = \overset{2}{\mathbf{w}}$, we obtain representation (15). Hence

$$\begin{aligned} &\overset{1}{\mathbf{w}} = \overset{1}{\mathbf{v}} + \overset{2}{\mathbf{v}}, \quad \theta = \varphi_1 + \varphi_2, \quad rot\overset{1}{\mathbf{w}} = 0, \quad div\overset{2}{\mathbf{w}} = 0, \\ &\Delta\overset{1}{\mathbf{v}} = 0, \qquad \Delta div\overset{1}{\mathbf{v}} = 0, \qquad (\Delta - s_1^2)div\overset{2}{\mathbf{v}} = 0, \qquad (\Delta - s_1^2)\overset{2}{\mathbf{v}} = 0, \\ &\Delta\varphi_1 = 0, \qquad (\Delta - s_1^2)\varphi_2 = 0, \qquad (\Delta - s_2^2)\overset{2}{\mathbf{w}} = 0. \end{aligned} \tag{18}$$

Substituting in (2),(3) $\mathbf{w} = \overset{1}{\mathbf{w}} + \overset{2}{\mathbf{w}}$ and replacing $\overset{1}{\mathbf{w}}$ and θ by their values from (17), we have

$$\begin{aligned} &k_7 s_1^2\overset{2}{\mathbf{v}} - k_2(\overset{1}{\mathbf{v}} + \overset{2}{\mathbf{v}}) = k_3 grad(\varphi_1 + \varphi_2), \\ &k\Delta\varphi_2 + k_1 div\overset{2}{\mathbf{v}} = 0. \end{aligned} \tag{19}$$

Equation(19) is satisfied by

$$\overset{1}{\mathbf{v}} = -\frac{k_3}{k_2}grad\varphi_1, \qquad \overset{2}{\mathbf{v}} = -\frac{k}{k_1}grad\varphi_2.$$

Finally, if we take

$$\overset{1}{\mathbf{v}} = -\frac{k_3}{k_2}grad\varphi_1, \qquad \overset{2}{\mathbf{v}} = -\frac{k}{k_1}grad\varphi_2$$

and they satisfy the conditions

$$\Delta\overset{1}{\mathbf{v}} = 0, \quad (\Delta - s_1^2)\overset{2}{\mathbf{v}} = 0,$$

then the general solution of the thermoelasticity equations (2),(3) takes the form

$$\begin{aligned} &\mathbf{w}(\mathbf{x}) = a\ grad\varphi_1(\mathbf{x}) + b\ grad\varphi_2(\mathbf{x}) + \overset{2}{\mathbf{w}}(\mathbf{x}), \\ &\theta(\mathbf{x}) = \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}), \quad a = -\frac{k_3}{k_2}, \qquad b = -\frac{k}{k_1}, \end{aligned} \tag{20}$$

where $\overset{2}{\mathbf{w}}$ satisfies the equations $(\Delta - s_2^2)\overset{2}{\mathbf{w}} = 0, \quad div\overset{2}{\mathbf{w}} = 0.$

Now let us prove that if the vector $\mathbf{W}(\mathbf{w},\theta) = 0$, then $\varphi_1 = \varphi_2 = 0, \quad \overset{2}{\mathbf{w}} = 0$. It follows from (20) that

$$a\ grad\varphi_1(\mathbf{x}) + b\ grad\varphi_2(\mathbf{x}) + \overset{2}{\mathbf{w}}(\mathbf{x}) = 0,$$

$$\varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}) = 0.$$

From here, after simple transformations we find that

$$div[a\ grad\varphi_1(\mathbf{x}) + b\ grad\varphi_2(\mathbf{x}) + \overset{2}{\mathbf{w}}(\mathbf{x})] = bs_1^2\varphi_2 = 0.$$

Thus we have $\varphi_1 = \varphi_2 = 0, \quad \overset{2}{\mathbf{w}} = 0$ and the proof is completed.

Let us assume that D^+ is a ball of radius R_1, centered at point $O(0,0,0)$ in space E_3 and S is a spherical surface of radius R_1.

Let us consider the metaharmonic equation

$$(\Delta + \nu^2)\psi = 0, \quad \nu \neq 0.$$

For this equation the following theorems are valid and we cite them without proof.

Lemma 1. *If regular vector $\boldsymbol{\psi}$ satisfies the conditions*

$$(\Delta + \nu^2)\boldsymbol{\psi} = 0, \quad \nu \neq 0, \quad div\boldsymbol{\psi} = 0,$$

$$(\boldsymbol{x}.\boldsymbol{\psi}) = 0, \quad \boldsymbol{x} \in D^+,$$

then it can be represented in the form

$$\boldsymbol{\psi}(\boldsymbol{x}) = [\boldsymbol{x}.\nabla]h(\boldsymbol{x})),$$

where

$$(\Delta + \nu^2)h(\boldsymbol{x}) = 0,$$

in addition if

$$\int\limits_{S(0,a_1)} h(\boldsymbol{x})ds = 0,$$

where $S(0,a_1) \subset D^+$ is an arbitrary spherical surface of radius a_1, then the function h in D^+ can be defined uniquely by means of vector ψ.

Lemma 2. *If regular vector $\boldsymbol{\psi}$ satisfies the conditions*

$$(\Delta + \nu^2)\boldsymbol{\psi} = 0, \quad \nu \neq 0 \quad div\boldsymbol{\psi} = 0, \quad \boldsymbol{x} \in D^+,$$

then it can be represented in the form

$$\boldsymbol{\psi}(\mathbf{x}) = [\mathbf{x}.\nabla]\varphi_3(\mathbf{x}) + rot[\mathbf{x}.\nabla]\varphi_4(\mathbf{x}),$$

where

$$(\Delta - s_2^2)\varphi_j = 0, \quad j = 3,4,$$

in addition if

$$\int\limits_{S(0,a_1)} \varphi_j ds = 0, \quad j = 3,4,$$

where $S(0,a_1) \subset D^+$ *is an arbitrary spherical surface of radius* a_1, *then the functions* $\varphi_j \quad j = 3,4$ *in* D^+ *can be defined uniquely by means of vector* ψ.

Lemma 1 and Lemma 2 are proved in [7].

Now from these theorems it follows that the following theorem is valid.

Theorem 3. *The regular solution* $\boldsymbol{W} = (\boldsymbol{w}, \theta)$, *where* $\boldsymbol{w} = (w_1, w_2, w_3)$, *of the homogeneous equations (2),(3), in* D^+, *can be represented in the form*

$$\begin{aligned} &\boldsymbol{w}(\boldsymbol{x}) = a\ grad\varphi_1(\boldsymbol{x}) + b\ grad\varphi_2(\boldsymbol{x}) + c\ rot\boldsymbol{\varphi}^{\mathbf{3}}(\boldsymbol{x}), \\ &\theta(\boldsymbol{x}) = \varphi_1(\boldsymbol{x}) + \varphi_2(\boldsymbol{x}), \end{aligned} \tag{21}$$

where

$$\Delta\varphi_1 = 0, \qquad (\Delta - s_1^2)\varphi_2 = 0, \qquad (\Delta - s_2^2)\boldsymbol{\varphi}^{\mathbf{3}} = 0, \qquad div\boldsymbol{\varphi}^{\mathbf{3}} = 0,$$

$$s_1^2 = \frac{kk_2 - k_1k_3}{kk_7} > 0, \qquad s_2^2 = \frac{k_2}{k_6} > 0, \qquad a = -\frac{k_3}{k_2}, \qquad b = -\frac{k}{k_1}, \qquad c = -\frac{k_6}{k_2},$$

$$\boldsymbol{\varphi}^{\mathbf{3}}(\mathbf{x}) = [\mathbf{x}\cdot\nabla]\varphi_3(\mathbf{x}) + rot[\mathbf{x}.\nabla]\varphi_4(\mathbf{x}), \qquad (\Delta - s_2^2)\varphi_j = 0, \qquad j = 3,4. \tag{22}$$

In addition if

$$\int\limits_{S(0,a_1)} \varphi_j ds = 0,$$

where $S(0,a_1) \subset D^+$ *is an arbitrary spherical surface of radius* a_1. *Between the vector* $\boldsymbol{W}(\boldsymbol{x}) = (\boldsymbol{w}, \theta)$ *and the functions* $\varphi_j, \quad j = 1,..,4$, *there exists one-to one correspondence.*

Remark. By virtue of the equality

$$rotrot[x.\nabla]\varphi_4 = -\Delta[x.\nabla]\varphi_4,$$

formula (21) can be rewritten in the form

$$\begin{aligned} &\mathbf{w}(\mathbf{x}) = a\ grad\varphi_1(\mathbf{x}) + b\ grad\varphi_2(\mathbf{x}) + [\mathbf{x}.\nabla]\varphi_4(\mathbf{x}) + c\ rot[\mathbf{x}.\nabla]\varphi_3(\mathbf{x}), \\ &\theta(\mathbf{x}) = \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}). \end{aligned} \tag{23}$$

Below we shall use solution (23) to solve the BVP for spherical ring.

Some auxiliary formulas

Let us introduce the spherical coordinates

$$x_1=\rho\sin\vartheta\cos\varphi,\quad x_2=\rho\sin\vartheta\sin\varphi,\quad x_3=\rho\cos\vartheta,\quad x\in\Omega,$$

$$y_1=R_1\sin\vartheta_0\cos\varphi_0,\quad y_2=R_1\sin\vartheta_0\sin\varphi_0,\quad y_3=R_1\cos\vartheta_0,\quad y\in S,\tag{24}$$

$$\rho^2=x_1^2+x_2^2+x_3^2,\quad 0\le\vartheta\le\pi,\quad 0\le\varphi\le 2\pi\quad 0\le\rho\le R_1.$$

In the sequel we use the following notations: If $\mathbf{g}(\mathbf{x})=\mathbf{g}(g_1,g_2,g_3)$ and $\mathbf{q}(\mathbf{x})=\mathbf{q}(q_1,q_2,q_3)$ then by symbols $(\mathbf{g}\cdot\mathbf{q})$ and $[\mathbf{g}\cdot\mathbf{q}]$ will be denote scalar product and vector product, respectively

$$(\mathbf{g}\cdot\mathbf{q})=\sum_{k=1}^{3}g_kq_k,\quad [\mathbf{g}\cdot\mathbf{q}]=(g_2q_3-g_3q_2,g_3q_1-g_1q_3,g_1q_2-g_2q_1).$$

The operator $\dfrac{\partial}{\partial S_k(x)}$ is determined as follows

$$[\mathbf{x}\cdot\nabla]_k=\frac{\partial}{\partial S_k(x)},\quad k=1,2,3,\qquad \nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right).$$

Simple calculations give

$$\frac{\partial}{\partial S_1(x)}=x_2\frac{\partial}{\partial x_3}-x_3\frac{\partial}{\partial x_2}=-cos\varphi ctg\vartheta\frac{\partial}{\partial\varphi}-sin\varphi\frac{\partial}{\partial\vartheta}$$

$$\frac{\partial}{\partial S_2(x)}=x_3\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_3}=-sin\varphi ctg\vartheta\frac{\partial}{\partial\varphi}+cos\varphi\frac{\partial}{\partial\vartheta}$$

$$\frac{\partial}{\partial S_3(x)}=x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}=\frac{\partial}{\partial\varphi}.$$

Below we use the following identities [7]

$$(\mathbf{x}.rotg(\mathbf{x}))=\sum_{k=1}^{3}\frac{\partial g_k(z)}{\partial S_k(z)},\quad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(z)}(rot[\mathbf{x}.\nabla]h)_k=0,$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(z)}(rot\mathbf{g}(\mathbf{x}))_k=\rho\frac{\partial}{\partial\rho}div\mathbf{g}(\mathbf{x})-\sum_{k=1}^{3}x_k\Delta\mathbf{g}_k(\mathbf{x}),\tag{25}$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}.\mathbf{g}]_k=\rho^2div\mathbf{g}(\mathbf{x})-(\mathbf{x}.\mathbf{g}(\mathbf{x}))-\rho\frac{\partial}{\partial\rho}(\mathbf{x}.\mathbf{g}(\mathbf{x})),$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(z)}[\mathbf{x}.rot\mathbf{g}(\mathbf{x})]_k=-(\rho\frac{\partial}{\partial\rho}+1)\sum_{k=1}^{3}\frac{\partial g_k(z)}{\partial S_k(z)},$$

$$\sum_{k=1}^{3}x_k\frac{\partial}{\partial S_k(x)}=0,\qquad \frac{\partial}{\partial S_k(x)}\frac{\partial}{\partial x_k}=\frac{\partial}{\partial x_k}\frac{\partial}{\partial S_k(x)},$$

$$\sum_{k=1}^{3}\frac{\partial^2}{\partial S_k^2(x)}=\frac{\partial^2}{\partial\vartheta^2}+ctg\vartheta\frac{\partial}{\partial\vartheta}+\frac{1}{sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2},\qquad \frac{\partial x_k}{\partial S_k}=0,$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}\frac{\partial}{\partial x_k}=0,\qquad \frac{\partial g(\rho)Y(\vartheta,\varphi)}{\partial S_k(x)}=g(\rho)\frac{\partial Y(\vartheta,\varphi)}{\partial S_k(x)}.$$

From this formulas it follows that, if g_m is the spherical harmonic, the operator $\dfrac{\partial}{\partial S_k}$, $k=1,2,3$, does not affect the order of the spherical function:

$$\sum_{k=1}^{3}\frac{\partial^2 g_m(\mathbf{x})}{\partial S_k^2(x)}=-m(m+1)g_m(\mathbf{x}).$$

We introduce the following notations:

$$(\mathbf{z}.\mathbf{f}^+)=h_1^+(\mathbf{z}),\quad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(z)}[\mathbf{z}.\mathbf{f}^+]_k=h_2^+(\mathbf{z}),\quad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(z)}f_k^+=h_3^+(\mathbf{z}),\quad f_4^+=h_4^+(\mathbf{z}).$$

$$(\mathbf{z}.\mathbf{f}^-)=h_1^-(\mathbf{z}),\quad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(z)}[\mathbf{z}.\mathbf{f}^-]_k=h_2^-(\mathbf{z}),\quad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(z)}f_k^-=h_3^-(\mathbf{z}),\quad f_4^-=h_4^+(\mathbf{z}).$$

Let us assume that $\mathbf{f}$ and f_4 are sufficiently smooth(differentiable) functions and h_k can be represented in the form

$$h_k^\pm(\mathbf{z})=\sum_{m=0}^{\infty}h_{km}^\pm(\vartheta,\varphi),$$

where $h_{km}^\pm$ is the spherical harmonic of order m :

$$h_{km}^\pm=\frac{2m+1}{4\pi R_1^2}\int\limits_S P_m(\cos\gamma)h_m^\pm(y)dS_y,$$

P_m is Legender polynomial of the m-th order, γ is an angle formed by the radius-vectors Ox and Oy,

$$\cos\gamma=\frac{1}{|x||y|}\sum_{m=1}^{3}x_ky_k.$$

The BVP for the spherical ring

Let us assume that Ω is a spherical ring, $R_1<|\mathbf{x}|<R_2$, centered at point $O(0,0,0)$ in the Euclidean 3D space E_3, S_1 is a spherical surface of radius R_1 and S_2 is a spherical surface of radius R_2. $S=S_1\cup S_2$.

The boundary value problem for the spherical ring is formulated as follows:

Find in the domain Ω a regular solution $\mathbf{U}(\mathbf{u},\mathbf{w},\theta)$ of equations (1),(2),(3) by the boundary conditions

$$(\mathbf{u})^- = \mathbf{F}^-(\mathbf{y}), \quad (\mathbf{w})^- = \mathbf{f}^-(\mathbf{y}), \quad \left(k\frac{\partial\theta}{\partial\mathbf{n}} + k_1\mathbf{n}\mathbf{w}\right)^- = f_4^-(\mathbf{y}), \quad \rho = R_1,$$

$$(\mathbf{u})^+ = \mathbf{F}^+(\mathbf{y}), \quad (\mathbf{w})^+ = \mathbf{f}^+(\mathbf{y}), \quad \left(k\frac{\partial\theta}{\partial\mathbf{n}} + k_1\mathbf{n}\mathbf{w}\right)^+ = f_4^+(\mathbf{y}), \quad \rho = R_2,$$

where $\mathbf{F}^{\pm}, \mathbf{f}^{\pm}, f_4^{\pm}$ are the given functions on S.

Theorem 4. *Two regular solutions of the considered BVP problem may differ by the vector* $\boldsymbol{V}(\boldsymbol{u},\boldsymbol{w},\theta)$, *where* $\mathbf{u}=0, \quad \mathbf{w}=0, \quad \theta = const.$

The general solution of the equations $(\Delta - s_k^2)\psi = 0, \quad k=1,2$, in the domain Ω has the form ([7])

$$\psi(\mathbf{x}) = \sum_{m=0}^{\infty}\left[\phi_m^{(2)}(is_k\rho)Y_m(\vartheta,\varphi) + \Psi_m^{(2)}(is_k\rho)Z_m(\vartheta,\varphi)\right], \quad R_1<\rho<R_2,$$

The general solution of the equation $\Delta\phi = 0$ in the domains Ω has the form

$$\phi(\mathbf{x}) = \sum_{m=0}^{\infty}\left[\frac{\rho^m}{(2m+1)R_2^{m-1}}Y_m(\vartheta,\varphi) + \frac{R_1^{m+2}}{(2m+1)\rho^{m+1}}Z_m(\vartheta,\varphi)\right], \quad R_1<\rho<R_2,$$

where $Y_m(\theta,\varphi), Z_m(\theta,\varphi)$ are the spherical harmonics,

$$\phi_m^{(2)}(is_k\rho) = \frac{\sqrt{R_2}J_{m+\frac12}(is_k\rho)}{\sqrt{\rho}J_{m+\frac12}(is_kR_2)}, \quad \Psi_m^{(2)}(is_k\rho) = \frac{\sqrt{R_1}H^{(1)}_{m+\frac12}(is_k\rho)}{\sqrt{\rho}H^{(1)}_{m+\frac12}(is_kR_1)}.$$

Using (23), we have

$$(\mathbf{x}\cdot\mathbf{w}) = a\rho\frac{\partial\varphi_1(\mathbf{x})}{\partial\rho} + b\rho\frac{\partial\varphi_2(\mathbf{x})}{\partial\rho} + c\sum_{k=1}^{3}\frac{\partial^2\varphi_3(\mathbf{x})}{\partial S_k^2(x)},$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{w}]_k = a\sum_{k=1}^{3}\frac{\partial^2\varphi_1(\mathbf{x})}{\partial S_k^2(x)} + b\sum_{k=1}^{3}\frac{\partial^2\varphi_2(\mathbf{x})}{\partial S_k^2(x)} - c\left(\rho\frac{\partial}{\partial\rho}+1\right)\sum_{k=1}^{3}\frac{\partial^2\varphi_3(\mathbf{x})}{\partial S_k^2(x)}, \tag{26}$$

$$\sum_{k=1}^{3}\frac{\partial w_k}{\partial S_k(x)} = \sum_{k=1}^{3}\frac{\partial^2\varphi_4(\mathbf{x})}{\partial S_k^2(x)}, \quad \theta(\mathbf{x}) = \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}).$$

Let the functions $\varphi_m(\mathbf{x}), \quad m=1,2,3,4$, be sought in the form

$$\varphi_1(\mathbf{x}) = \sum_{m=0}^{\infty}\left[\frac{\rho^m}{(2m+1)R_2^{m-1}}Y_{1m}(\vartheta,\varphi) + \frac{R_1^{m+2}}{(2m+1)\rho^{m+1}}Z_{1m}(\vartheta,\varphi)\right],$$

$$\varphi_2(\mathbf{x}) = \sum_{m=0}^{\infty} \left[\phi_m^{(2)}(is_1\rho)Y_{2m}(\vartheta,\varphi) + \Psi_m^{(2)}(is_1\rho)Z_{2m}(\vartheta,\varphi)\right],$$

$$\varphi_j(\mathbf{x}) = \sum_{m=0}^{\infty} \left[\phi_m^{(2)}(is_2\rho)Y_{jm}(\vartheta,\varphi) + \Psi_m^{(2)}(is_2\rho)Z_{jm}(\vartheta,\varphi)\right], j=3,4,$$

The conditions $\int\limits_{S(0,a_1)} \varphi_j ds = 0 \quad j=3,4$ in fact mean that

$$Y_{30} = Y_{40} = 0, \qquad Z_{30} = Z_{40} = 0.$$

Substitute in (26) the functions $\varphi_j(\mathbf{x})$, passing to the limit as $\rho \to R_1$, $\rho \to R_2$ and taking into account boundary conditions , for determining the unknown values Y_{jm} and Z_{jm}, we obtain the following system of algebraic equations

$$\frac{maR_1^m}{(2m+1)R_2^{m-1}}Y_{1m} - \frac{(m+1)aR_1}{2m+1}Z_{1m} + b\left[\rho\frac{\partial}{\partial\rho}\phi_m^{(2)}(is_1\rho)\right]_{\rho=R_1} Y_{2m}$$
$$+b\left[\rho\frac{\partial}{\partial\rho}\Psi_m^{(2)}(is_1\rho)\right]_{\rho=R_1} Z_{2m} - cm(m+1)\left\{\left[\phi_m^{(2)}(is_1\rho)\right]_{\rho=R_1} Y_{3m} + Z_{3m}\right\} = h_{1m}^-,$$
$$\frac{maR_2}{(2m+1)}Y_{1m} - \frac{(m+1)aR_1^{m+2}}{(2m+1)R_2^{m+1}}Z_{1m} + b\left[\rho\frac{\partial}{\partial\rho}\phi_m^{(2)}(is_1\rho)\right]_{\rho=R_2} Y_{2m}$$
$$+b\left[\rho\frac{\partial}{\partial\rho}\Psi_m^{(2)}(is_1\rho)\right]_{\rho=R_2} Z_{2m} - cm(m+1)\left\{Y_{3m} + \left[\Psi_m^{(2)}(is_2\rho)\right]_{\rho=R_2} Z_{3m}\right\} = h_{1m}^+,$$
$$-\frac{m(m+1)aR_1^m}{(2m+1)R_2^{m-1}}Y_{1m} - \frac{am(m+1)R_1}{2m+1}Z_{1m} - bm(m+1)\left\{\phi_m^{(2)}(is_1R_1)Y_{2m} + Z_{2m}\right\}$$
$$+cm(m+1)\left\{\left[(\rho\frac{\partial}{\partial\rho}+1)\phi_m^{(2)}(is_2\rho)\right]_{\rho=R_1} Y_{3m} + \left[(\rho\frac{\partial}{\partial\rho}+1)\Psi_m^{(2)}(is_2\rho)\right]_{\rho=R_1} Z_{3m}\right\} = h_{2m}^-,$$
$$-\frac{m(m+1)aR_2}{2m+1}Y_{1m} - \frac{am(m+1)R_1^{m+2}}{(2m+1)R_2^{m+1}}Z_{1m} - bm(m+1)\left\{Y_{2m} + \Psi_m^{(2)}(is_1R_2)Z_{2m}\right\}$$
$$+cm(m+1)\left\{\left[(\rho\frac{\partial}{\partial\rho}+1)\phi_m^{(2)}(is_2\rho)\right]_{\rho=R_2} Y_{3m} + \left[(\rho\frac{\partial}{\partial\rho}+1)\Psi_m^{(2)}(is_2\rho)\right]_{\rho=R_2} Z_{3m}\right\} = h_{2m}^+,$$

$$-m(m+1)\{\Phi_m^{(2)}(is_2R_1)Y_{4m} + Z_{4m}\} = h_{3m}^-,$$

$$-m(m+1)\{Y_{4m} + \Psi_m^{(2)}(is_2R_2)Z_{4m}\} = h_{3m}^+, \tag{27}$$

$$\frac{mR_1^{m-1}}{(2m+1)R_2^{m-1}}Y_{1m} - \frac{m+1}{2m+1}Z_{1m} + \left[\frac{\partial}{\partial\rho}\phi_m^{(2)}(is_1\rho)\right]_{\rho=R_1} Y_{2m}$$

$$+\left[\frac{\partial}{\partial\rho}\Psi_m^{(2)}(is_1\rho)\right]_{\rho=R_1} Z_{2m}=\frac{h_{4m}^-}{k}+\frac{1}{R_1 b}h_{1m}^-,$$

$$\frac{m}{2m+1}Y_{1m}-\frac{(m+1)R_1^{m+2}}{(2m+1)R_2^{m+2}}Z_{1m}+\left[\frac{\partial}{\partial\rho}\phi_m^{(2)}(is_1\rho)\right]_{\rho=R_2} Y_{2m}$$

$$+\left[\frac{\partial}{\partial\rho}\Psi_m^{(2)}(is_1\rho)\right]_{\rho=R_2} Z_{2m}=\frac{h_{4m}^+}{k}+\frac{1}{R_2 b}h_{1m}^+.$$

Note that for $m=0$, (27) is transformed to the system

$$Z_{10}=\frac{b}{k(a-b)}h_{40}^-,\quad 0=h_{20}^+,\quad 0=h_{20}^-,\quad 0=h_{30}^+,\quad 0=h_{30}^-,$$

$$0\cdot Y_{10}+\left[\frac{\partial}{\partial\rho}\phi_0^{(2)}(is_1\rho)\right]_{\rho=R_1} Y_{20}+\left[\frac{\partial}{\partial\rho}\Psi_0^{(2)}(is_1\rho)\right]_{\rho=R_1} Z_{20}=\frac{h_{10}^-}{bR_1}+\frac{a}{b}Z_{10}, \tag{28}$$

$$0\cdot Y_{10}+\left[\frac{\partial}{\partial\rho}\phi_0^{(2)}(is_1\rho)\right]_{\rho=R_2} Y_{20}+\left[\frac{\partial}{\partial\rho}\Psi_0^{(2)}(is_1\rho)\right]_{\rho=R_2} Z_{20}=\frac{h_{10}^+}{bR_2}+\frac{aR_1^2}{bR_2^2}Z_{10}.$$

Taking into account the identities $J_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\sin z$, $\quad H_{\frac{1}{2}}^{(1)}(z)=-i\sqrt{\frac{2}{\pi z}}\exp(iz)$, after certain calculations, the determinant of system (28) takes the form

$$\delta=\frac{\exp R_1 s_1}{R_1 R_2\sinh s_1 R_2}\{(s_1^2 R_1 R_2-1)\sinh s_1(R_2-R_1)$$

$$+s_1(R_2-R_1)\cosh s_1(R_2-R_1)\}\neq 0.$$

Thus we have shown that Y_{10} is an arbitrary constant and for the solution to exist it is necessary that the conditions $h_{20}^+=0,\quad h_{20}^-=0,\quad R_2^2h_{40}^+=R_1^2h_{40}^-$ be fulfilled. By virtue of the uniqueness theorems of solutions of the BVP, we conclude that the determinant of system (26) for $m\geq 1$ is different from zero and we obtain the required solution of problem in the form of series.

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Author's address:

L. Bitsadze
I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: lamarabits@yahoo.com