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ON LINEAR BOUNDARY VALUE PROBLEMS FOR MULTIDIMENSIONAL REGULAR DIFFERENCE SYSTEMS

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Abstract. The Green's type theorems are established for unique solvability of linear boundary value problems for multidimensional systems of linear regular difference equations. Moreover, a successive approximation method is given for the construction of the solution of the difference system under the Cauchy condition.

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1. Statement of the problem and formulation of the results

This work is dedicated to the investigation of the solvability question of the regular difference system

$$\Delta y(k-1) = G_1(k)y(k-1) + G_2(k)y(k) + g(k) \quad (k = 1, 2, ...) \tag{1.1}$$

under the general boundary value problem

$$\mathcal{L}(y) \equiv \sum_{i=1}^{\infty} L(k)y(k) = c_0, \tag{1.2}$$

where $G_j \in E(\mathbb{N}_0, \mathbb{R}^{n\times n})$ $(j = 1, 2)$, $L \in E(\mathbb{N}_0, \mathbb{R}^{n\times n})$, $\mathcal{L} : BV_{\cup}(\mathbb{N}_0\mathbb{R}^n) \to \mathbb{R}^n$ is a bounded linear operator, and $g \in E(\mathbb{N}_0, \mathbb{R}^n)$ are respectively, discrete matrix and vector functions, and $c_0 \in \mathbb{R}^n$. In this work the Green's type theorem is proved for the unique solvability of the problem (1.1),(1.2) in the case when $G_j \in E(\mathbb{N}_0, \mathbb{R}^{n\times n})$ $(j = 1, 2)$, $L \in E(\mathbb{N}_0, \mathbb{R}^{n\times n})$ and $g(k) \in E(\mathbb{N}_0, \mathbb{R}^n)$ are, respectively, so called regular matrix and vector functions on the set $\mathbb{N}_0$(see below). Moreover, successive approximations methods is investigated for constructing the solution for the Cauchy problem for the system (1.1). For investigating this problem we use the theory of so called generalized ordinary differential equations [1]. Analogous questions for the finite difference system are investigated in [1,2].

Along with the problem (1.1),(1.2) we consider the corresponding homogeneous problem

$$\Delta y(k-1) = G_1(k)y(k-1) + G_2(k)y(k) \quad (k = 1, 2, ...), \tag{1.1_0}$$

$$\mathcal{L}(y) = 0. \tag{1.2_0}$$

Throughout the paper, the following notation and definitions will be used.

$\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, \ldots\}$.

$\mathbb{R} =]-\infty,+\infty[$, $[a,b]$ and $]a,b[$ $(a, b \in$ are, respectively, a closed and an open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ – matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max\left\{\sum_{i=1}^{n}|x_{ij}| : j = 1,\dots,m\right\}.$$

If $X = (x_{ij})_{i,j=1}^{n,m}$, then $|X| = (|x_{ij}|)_{i,j=1}^{n,m}$.

$O_{n\times m}$ is the zero $n\times m$-matrix.

$\mathbb{R}_+^{n\times m} = \{(x_{i,j})_{i,j=1}^{n,m} : x_{i,j} \ge 0 \quad (i = 1,\dots,n; j = 1,\dots,m)\}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1} is the matrix, inverse to X; $\det X$ is the determinant of X; and $r(X)$ is the spectral radius of X.

I_n is the identity $n\times n$-matrix.

$E(\mathbb{N}_0,\mathbb{R}^{n\times m})$ is the set of all matrix-functions $Y : \mathbb{N}_0 \to \mathbb{R}^{n\times m}$.

Δ is the difference operator of the first order, i.e.,

$$\Delta Y(k-1) = Y(k) - Y(k-1) \ \text{ for } \ Y \in E(\mathbb{N}_0,\mathbb{R}^{n\times m}) \ \ (k = 1,2,...).$$

We say that the discrete matrix function $X \in E(\mathbb{N}_0,\mathbb{R}^{l\times m})$ has the bounded total variation on the set $\mathbb{N}_0$ if

$$\sum_{k=1}^{\infty}\|\Delta X(k-1)\| < +\infty.$$

In this case we assume

$$\|X\|_v = \sum_{k=1}^{\infty}\|\Delta X(k-1)\|.$$

By $BV_v(\mathbb{N}_0;\mathbb{R}^{n\times m})$ we denote the Banach space of all discrete matrix-functions $E(\mathbb{N}_0,\mathbb{R}^{n\times m})$ with the norm $\|.\|_v$.

The inequalities between the matrices are understood componentwise.

A matrix function is said to be continuous, integrable, nondecreasing, etc., if such is every its component.

Under a solution of the difference problem (1.1),(1.2) we understand a matrix function $y \in BV_v(\mathbb{E}_0,\mathbb{R}^n)$ satisfying difference system (1.1) (i.e., the equality (1.1) for every $k \in \mathbb{N}$) and the boundary condition (1.2).

Below we show that, in the regular case, i.e., when discrete matrix G_1 and G_2 and vector g functions are regular, every discrete vector-function $y \in E(\mathbb{N}_0,\mathbb{R}^n)$ satisfying difference system (1.2) belongs to $BV_v(\mathbb{E}_0,\mathbb{R}^n)$, as well. So that the definition of solutions of system (1.1) given above, is natural for the regular case.

The discrete matrix-function $X \in E(\mathbb{N}_0,\mathbb{R}^{n\times m})$ is said to be regular if

$$\sum_{k=1}^{\infty}\|X(k)\| < +\infty.$$

Definition 1.1. The system (1.1) is called regular if the matrix-and vector functions G_1, G_2 and g are regular, i.e., (1.3)

$$\sum_{k=1}^{\infty} \|G_j(k)\| < +\infty \quad (j = 1, 2) \tag{1.3}$$

and

$$\sum_{k=1}^{\infty} \|g(k)\| < +\infty. \tag{1.4}$$

We will assume that system (1.1) is regular. Moreover, we assume that the matrix function $L \in E(\mathbb{N}_0, \mathbb{R}^{n\times n})$ is regular, too.

Let Y be the fundamental matrix of the system (1.1_0) under the condition

$$Y(0) = I_n.$$

If the condition

$$\det\big(I_n + (-1)^j G_j(k)\big) \neq 0 \;\text{ for }\; k \in \{1, 2, ...\} \;\; (j = 1, 2) \tag{1.5}$$

is valid, then the fundamental matrix Y of the system (1.1_0) exists and

$$Y(k) = \prod_{l=k}^{0} \big(I_n - G_1(l)\big)^{-1}\big(I_n + G_2(l)\big) \;\text{ for }\; k \in \{1, 2, ...\}. \tag{1.6}$$

We assume

$$D = \sum_{l=0}^{\infty} L(l)Y(l) \;\text{ and }\; D(j) = \sum_{l=0}^{j} L(l)Y(l) \;\; (j = 0, 1, ...). \tag{1.7}$$

If

$$\det D \neq 0, \tag{1.8}$$

then we assume

$$\mathcal{G}(k,j) = \begin{cases} Y(k)D^{-1}D(j-1)Y^{-1}(j)\big(I_n - G_1(j)\big)^{-1} & \text{for } 0 \le j < k, \\ -Y(k)\big(I_n - D^{-1}D(j-1)\big)Y^{-1}(j)\big(I_n - G_1(j)\big)^{-1} & \text{for } 0 \le k < j, \\ O_{n\times n} & \text{for } k = j, \end{cases} \tag{1.9}$$

where $Y(k)$ is the fundamental matrix of the system (1.1_0) defined by (1.6). The matrix function $\mathcal{G}(k,j)$ is called the Green matrix of the problem (1.1_0),(1.2_0).

Theorem 1.1. *Let the condition (1.5) hold and let the system (1.1) be regular. Then the boundary value problem (1.1),(1.2) has a unique solution if and only if the corresponding homogeneous problem* $(1.1_0), (1.2_0)$ *has only the trivial solution. If the letter condition holds, then the solution* y *of problem (1.1),(1.2) admits the representation*

$$y(k) = Y(k)D^{-1}c_0 + \sum_{l=1}^{\infty} \mathcal{G}(k,l)g(l) \;\; \textit{for} \;\; k \in \mathbb{N}_0, \tag{1.10}$$

where $\mathcal{G}(k,l)$ *is the Green matrix of the problem* $(1.1_0), (1.2_0)$.

Remark 1.1. We note the homogeneous problem $(1.1_0), (1.2_0)$ has only the trivial solution (as well problem (1.1),(1.2) is uniquely solvable) if and only if the condition (1.8) is valid. Therefore, there exist the Green matrix appearing in Theorem 1.1.

Remark 1.2. If the condition (1.8) is not fulfilled, then for every regular $g \in E(\mathbb{N}_0, \mathbb{R}^n)$ there exists a vector $c_0 \in \mathbb{R}^n$ such that problem (1.1),(1.2) has no solution. In addition, if $\mathcal{L} : E(\mathbb{N}_0, \mathbb{R}^n) \to \mathbb{R}^n)$ is the onto mapping, then for every $c_0 \in \mathbb{R}^n$ there exists a regular function $g \in E(\mathbb{N}_0, \mathbb{R}^n)$ such that the problem (1.1),(1.2) is not solvable.

We give a successive approximation method of construction of the solution of the system (1.1), too, under the Cauchy condition

$$y(k_0) = c_0, \tag{1.11}$$

where $k_0 \in \mathbb{N}$, $c_0 \in \mathbb{R}^n$.

Theorem 1.2 *Let*

$$\det\big(I_n + (-1)^j G_j(k)\big) \neq 0 \ \ \textit{for} \ \ (-1)^j (k - k_0) < 0 \ \ (j = 1, 2). \tag{1.12}$$

Then the Cauchy problem (1.1),(1.11) has a unique solution $y \in E(\mathbb{N}, \mathbb{R}^n)$ *and*

$$\lim_{m\to\infty} y_m(k) = y(k) \ \ \textit{uniformly for} \ \ k \in \mathbb{N}_0, \tag{1.13}$$

where

$$y_m(k_0) = c_0 \ \ (m = 0, 1, ...),$$
$$y_0(k) = \big(I_n + (-1)^j G_j(k + j - 1)\big)^{-1} c_0 \ \ \textit{for} \ \ (-1)^j (k - k_0) < 0 \ \ (j = 1, 2)$$

and

$$y_m(k) = \big(I_n + (-1)^j G_j(k + j - 1)\big)^{-1} \Bigg[c_0 + (-1)^j G_j(k + j - 1) y_{m-1}(k)$$
$$-(-1)^j \sum_{l=k_0+1+(j-1)(k-k_0)}^{k-(j-1)(k-k_0)} \big(G_1(l) y_{m-1}(l) + G_2(l) y_{m-1}(l - 1)\big) \Bigg]$$
$$\textit{for} \ \ (-1)^j (k - k_0) < 0 \ \ (j = 1, 2).$$

2. Generalized differential equations

We give some necessary definition to formulate bases of the theory of the generalized ordinary differential equations.

The interest in the theory of generalized ordinary differential equations has also been stimulated to a considerable extent by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see, e.g. [1-10] and the references therein).

If $X : [a,b] \to \mathbb{R}^{n\times m}$ is a matrix-function, then $V_a^b(X)$ is the sum of total variations on $[a,b]$ of its components x_{ij} $(i = 1,\dots,n; j = 1,\dots,m)$; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) = V_a^t(x_{ij})$ for $a < t \le b$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t $(X(a-) = X(a)$, $X(b+) = X(b))$.

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the Banach space of all bounded variation matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$ (i.e., such that $V_a^b(X) < \infty$) with the norm $\|X\|_v = \|X(a)\| + V_a^b(X)$.

$s_j : \mathrm{BV}([a,b],\mathbb{R}) \to \mathrm{BV}([a,b],\mathbb{R})$ $(j = 0,1,2)$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,$$

$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \ \text{ and } \ s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \ \text{ for } \ a < t \le b,$$

and

$$s_0(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \ \text{ for } \ t \in [a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,ds_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_0(g))$, corresponding to the function $s_0(g)$.

If $a = b$, then we assume $\int_a^b x(t)\,dg(t) = 0$, and if $a > b$, then we assume $\int_a^b x(t)\,dg(t) = -\int_b^a x(t)\,dg(t)$.

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_s^t x(\tau)\,dg_1(\tau) - \int_s^t x(\tau)\,dg_2(\tau) \ \text{ for } \ s \le t.$$

If $G = (g_{ik})_{i,k=1}^{l,n} \in \mathrm{BV}([a,b],\mathbb{R}^{l\times n})$ and $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to \mathbb{R}^{n\times m}$, then

$$S_j(G)(t) \equiv (s_j(g_{ik})(t))_{i,k=1}^{l,n} \quad (j = 0,1,2)$$

and

$$\int_a^b dG(\tau)\cdot X(\tau) = \left(\sum_{k=1}^{n} \int_a^b x_{kj}(\tau)\,dg_{ik}(\tau)\right)_{i,j=1}^{l,m}.$$

Let $A \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ and $f \in \mathrm{BV}([a,b],\mathbb{R}^n)$.

Under a solution of the system of linear generalized ordinary differential equations

$$dx(t) = dA(t)\cdot x(t) + df(t) \tag{2.1}$$

we understand a vector-function $x \in \mathrm{BV}([a,b],\mathbb{R}^n)$ such that

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot x(\tau) + f(t) - f(s) \ \text{ for } \ a \le s < t \le b.$$

We consider system (2.1) with the boundary value condition

$$\ell(x) = c, \tag{2.2}$$

where $\ell : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n)$ is a linear bounded operator, and $c \in \mathbb{R}^n$ is a constant vector.

The question of the unique solvability of the generalized boundary value problem (2.1),(2.2) is investigated in [1,2,10] (see also the references therein).

3. Proof of the results

We will rewrite problem (1.1),(1.2) in the form of problem (2.1),(2.2) in order to apply the results from [1,2,10] to the last generalized problem.

Let Y be the fundamental matrix of system (1.1) under the condition $Y(0) = I_n$. Then by (1.3) and (1.6) there exists a positive number $r > 0$ such that

$$\|Y(k)\| < r \ \text{ for } \ k \in \mathbb{N}_0. \tag{3.1}$$

We assume

$$G_j(0) = O_{n\times n} \ \ (j = 1, 2), \ \ g(0) = 0_n.$$

Let $y \in E(\mathbb{N}_0, \mathbb{R}^n)$ be an arbitrary solution of the problem (1.1),(1.2) and let $z = (z_i)_{i=1}^2$, where $z_i \in E(\mathbb{N}_0, \mathbb{R}^n)(i = 1, 2)$ be functions, defined by

$$z_1(k) = z_2(k) = y(k) \ \ (k = 0, 1, ...).$$

Then by (3.1) we get

$$\|y(k)\| < r\|y(0)\| \ \text{ for } \ k \in \mathbb{N}_0.$$

From this by (1.1),(1.3) and (1.4) we have

$$\sum_{k=0}^{\infty} \|\Delta y(k-1)\| < +\infty$$

and

$$\sum_{k=0}^{\infty} \|\Delta z(k-1)\| < +\infty. \tag{3.2}$$

Moreover, it is evident that

$$\Delta \begin{pmatrix} z_1(k-1) \\ z_2(k-1) \end{pmatrix} = \begin{pmatrix} G_1(k)z_1(k) + G_2(k)z_2(k-1) + g(k) \\ G_1(k)z_1(k) + G_2(k)z_2(k-1) + g(k) \end{pmatrix} \ \text{ for } \ k \in \mathbb{N}_0 \tag{3.3}$$

and

$$\zeta_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_0 \\ 0 \end{pmatrix}, \ \ \zeta_2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.4}$$

where

$$\zeta_1\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \equiv \begin{pmatrix} \mathcal{L}(z_1) \\ 0 \end{pmatrix},$$

and $\zeta_2 : E(\mathbb{N}_0, \mathbb{R}^{2n}) \to \mathbb{R}^{2n}$ is an arbitrary operator such that the condition $\zeta_2(z) = 0$ guarantees the equality $z_1(k_0) = z_2(k_0)$ for some $k_0 \in \mathbb{N}_0$.

We will assume that

$$\zeta_2\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2(k_0) - z_1(k_0) \\ z_2(k_0) - z_1(k_0) \end{pmatrix},$$

where k_0 is an arbitrary fixed integer from $\mathbb{N}_0$.

The contrary is evident too. If the vector-function $z = (z_i)_{i=1}^2$ is a solution of problem (3.1),(3.2) then $z_1(k) \equiv z_2(k)$ and this discrete vector function is a solution of problem (1.1),(1.2). Therefore, problems (1.1),(1.2) and (3.3),(3.4) are equivalent among themselves.

We note that by (1.3) there exists $k_0 \in \mathbb{N}$ such that $\|G_j(k_0)\| < 1/2$ $(j = 1, 2)$ and, therefore, the inverse matrices $\big(I_n + (-1)^j G_j(k)\big)^{-1}(j = 1, 2)$ exist for $k \ge k_0$. From this, taking into account the condition (1.3) we get that there exists a constant $r_1 > 0$ such that

$$\|\big(I_n + (-1)^j G_j(k)\big)^{-1}\| < r_1 \ \text{ for } \ k \ge k_0 \ \ (j = 1, 2). \tag{3.5}$$

Let now

$$I_{1k} = [t_k, t_{k+1}[\ \ \textit{and} \ \ I_{1k} =]t_k, t_{k+1}] \ \text{ for } \ k \in \mathbb{N}_0,$$

where $t_k = k/(k + 1)$ $(k = 0, 1, ...)$.

Let $x = (x_i)_{i=1}^2$ be a vector function defined by

$$x_i(t) = z_i(k) \ \text{ for } \ t \in I_{ik} \ \ (i = 1, 2; k = 0, 1, ...). \tag{3.6}$$

Then by (3.2) we have $x \in \mathrm{BV}([0, 1], \mathbb{R}^{2n})$.

It is not difficult to verify that the vector function x will be a solution of the 2n-dimension problem (2.1),(2.2) with $a = 0, b = 1$,

$$A(t) \equiv (A_{ij}(t))_{i,j=1}^2, \tag{3.7}$$

$$A_{ij}(t) = \sum_{l=0}^{k} G_j(l) \ \text{ for } \ t \in I_{ik} \ \ (i, j = 1, 2; k = 0, 1, ...); \tag{3.8}$$

$$f(t) \equiv (f_i(t))_{i=1}^2, \tag{3.9}$$

$$f_i(t) = \sum_{l=0}^{k} g(l) \ \text{ for } \ t \in I_{ik} \ \ (i, j = 1, 2; k = 0, 1, ...); \tag{3.10}$$

$$\ell(x) = \big(\zeta_i(z)\big)_{i=1}^2 \ \text{ for } \ x = (x_i)_{i=1}^2, \ \ x_i \in \mathrm{BV}([0, 1], \mathbb{R}^n), \ (i = 1, 2) \tag{3.11}$$

and

$$c = \begin{pmatrix} c_0 \\ 0 \end{pmatrix}.$$

It is evident that the inverse proposition is true as well. So that the following lemma is true.

Lemma 1.1 *Let $y \in E(\mathbb{N}_0, \mathbb{R}^n)$ be a solution of problem (1.1),(1.2). Then the vector function $x = (x_i)_{i=1}^n \mathrm{BV}([0,1], \mathbb{R}^n)$, defined by (3.6), will be a solution of the $2n$-dimensional generalized boundary value problem (2.1),(3.2), where $a = 0, b = 1$, and matrix-and vector functions A and f, linear operator ℓ and constant vector c are defined, respectively, by (3.7)-(3.11). On the contrary, if the vector-function $x = (x)_{i=1}^n \in BV([0,1], \mathbb{R}^{2n})$ is a solution of the last $2n$-dimensional problem (2.1),(3.2), then the vector-function $y \in E(\mathbb{N}_0, \mathbb{R}^n)$, $y(k) \equiv z_1(k)$, will be a solution of the problem (1.1),(1.2), where*

$$G_i(k) \equiv \Delta A_{1i}(k) \ \ (i = 1, 2), \ \ g(k) \equiv \Delta f_1(k),$$

and $\mathcal{L}(y)$ and c_0 are n-vectors whose i-th component coincides with i-th component of $\ell(y)$ and c, respectively, for every $i \in \{1, ..., n\}$.

Using the lemma we conclude that the theorems and remarks immediately follow from corresponding results of paper [1,2,10].

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