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SOLUTION OF THE PROBLEMS OF ELASTOSTATICS FOR DOUBLE POROUS AN ELASTIC PLANE WITH A CIRCULAR HOLE. THE UNIQUENESS THEOREMS

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Abstract. In the present paper we solve explicitly, by means of absolutely and uniformly convergent series, the second boundary value problems of porous elastostatics for the plane with a circular hole.

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1. Introduction

In the E.C. Aifantis theory of consolidation the elastic medium with double porosity is considered. For such a kind of media the problem is formulated under the following boundary conditions: the value of the displacement (or stress) vector and the value of pressures (or normal derivative pressures) of a liquid in pores are given. In the present work we solve explicitly, by means of absolutely and uniformly convergent series, the second boundary value problem of porous elastostatics for the plane with a circular hole. From the point of view of applications, very actual is the construction of solutions explicitly which allows one to perform numerical analysis of the problem under investigation.

2. Basic equations

We consider the plane D with a circular hole. Let R be the radius of the boundary S. Find a regular vector $U(u(x), p_1(x), p_2(x))$, satisfying in D a system of equations [1,2]:

$$\begin{aligned} &\mu\Delta(u(x)) + (\lambda+\mu) graddiv(u(x)) = grad[\beta_1 p_1(x) + \beta_2 p_2(x)],\\ &(m_1\Delta - k)p_1(x) + kp_2(x) = 0, \\ &kp_1(x) + (m_2\Delta - k)p_2(x) = 0, x \in D \end{aligned} \tag{1}$$

and on the circumference S one of the following conditions:

$$\begin{aligned} &I.u(z) = f(z), \quad \partial_n p_1 = f_3(z), \quad \partial_n p_2(z) = f_4(z);\\ &II.P(\partial_z, n)U(z) = f(z), \quad p_1(z) = f_3(z), \quad p_2(z) = f_4(z), \end{aligned} \tag{2}$$

where $\lambda, \mu, m_1, m_2, \beta_1, \beta_2$ are the known elastic and physical constants, $k, m_i > 0, i = 1,2[1,2]$; $u(x) = (u_1(x)), u_2(x))$ is the displacement of the point x;$n(z) = (n_1(z), n_2(z))$, $z = (z_1, x_2) \in S$, p_1 is the fluid pressure within the primary pores and p_2 is the fluid pressure within the secondary pores; Δ is the Laplace operator; $f(z) = (f_1(z), f_2(z))$, $f_3(z), f_4(z)$ are the given functions on the circumference S;

$$P(\partial_x, n)U(x) = T(\partial_x, n)u(x) - n(x)[\beta_1 p_1(x) + \beta_2 p_2(x)] \tag{3}$$

is the stress vector of the theory of poroelasticity; $T(\partial_x, n)u(x) = \mu\partial_n u(x) + \lambda n(x)div(u(x)) + \mu \sum_{i=1}^{\infty} n_i(x)gradu_i(x)$ is the stress vector of the theory of elasticity; $\partial_n = \frac{\partial}{\partial n}$; $\partial_k = \frac{\partial}{\partial x_k}$, $k = 1, 2$.

Vector $U(x)$ satisfies the following conditions at infinity:

$$U(x) = O(1), \quad \partial_k U(x) = O(1), \quad k = 1, 2. \tag{4}$$

We will study separately the following problems:

1. Find in a plane D solution $u(x)$ of equation $(1)_1$, if on the circumference S there are given the values: a) of the vector u - problem A_1; b) of the vector $P(\partial_z, n)u(z)$ - problem A_2.

2. Find in a plane D solutions $p_1(x)$ and $p_2(x)$ of the system of equations $(1)_2$ and $(1)_3$, if on the circumference S there are given the values: a) of the function p_1 and the vector p_2 - problem B_1; b)of the derivates $\partial_n p_1(z)$ and $\partial_n p_2(z)$ - problem B_2.

Thus the above-formulated problems of poroelastostatics can be considered as a union of two problems: I - (A_1, B_2) and II - (A_2, B_1).

3. Uniqueness theorems

For regular solutions of equation $(1)_1$ and equations $(1)_2$ and $(1)_3$ Green's formulas:

$$\int_D [E(u(x), u(x)) - (\beta_1 p_1(x) + \beta_2 p_2)(x)divu(x)]dx = \int_S u(y)P(\partial_y, n(y))d_y S; \tag{5}$$

$$\int_D [m_1 \mid gradp_1 \mid^2 + m_2 \mid gradp_2 \mid^2 + k(p_2 - p_1)^2]dx$$
$$= \int_S [m_1 p_1(y)\partial_n p_1(y) + m_2 p_2(y)\partial_n p_2(y)]d_y S \tag{6}$$

are valid, where

$$E(u, u) = (\lambda + \mu)(\partial_1 u_1 + \partial_2 u_2)^2 + \mu(\partial_1 u_1 - \partial_2 u_2)^2 + \mu(\partial_2 u_1 + \partial_1 u_2)^2$$

is a nonnegative quadratic form under the condition that $\lambda + \mu > 0$, $\mu > 0$.

Problems B. Since m_i, $k > 0$, therefore in the case of homogeneous boundary conditions (2) the product $p_i\partial_n p_i$ vanishes. Let p_1 and p_2 be differences of two different solutions of problems B_1 and B_2. By virtue of equality (6), the following theorems are valid.

Theorem 1. *The difference of two arbitrary solutions of problem B_1 is equal to zero: $p_1(x) = p_2(x) = 0$.*

Theorem 2. *The difference of two arbitrary solutions of problem B_2 may differ only by an arbitrary constant $p_1(x) = p_2(x) = c$.*

Problems A. Let (u', p_1', p_2') and (u'', p_1'', p_2'') be two different solutions of any of the problems I, II. Then the differences $u = u' - u''$, $p_1 = p_1' - p_1''$ and $p_2 = p_2' - p_2''$ are the solutions of the corresponding homogeneous problems.

Taking into account Theorems 1 and 2, and formula (5), under the homogeneous boundary conditions for the problems I and II, we obtain $E(u, u) = 0$. The solution of the above equation has the form

$$u_1(x) = -cx_2 + q_1, \quad u_2(x) = cx_1 + q_2, \tag{7}$$

where c, q_1 and q_2 are arbitrary constants.

Taking into account conditions (4) and formulas (7), we obtain:

$u_1(x) = u_2(x) = 0$ - for problem A_1;

$u_1(x) = q_1, \quad u_2(x) = q_2$ - for problem A_2;

The following theorems are valid.

Theorem 3. *The difference of two arbitrary solutions of problem I is the vector $U(u_1(x), u_2(x), p_1(x), p_2(x))$, where $u_1 = u_2 = 0$, $p_1 = p_2 = c$;*

Theorem 4. *The difference of two arbitrary solutions of problem II is the vector $U(u_1(x), u_2(x), p_1(x), p_2(x))$, where $u_1(x) = q_1, \quad u_2(x) = q_2$ and $p_1 = p_2 = 0$.*

4. Solutions of the problems

On the basis of the system $[(1)_2, (1)_3]$, we can write $m_1 m_2 \triangle(\triangle + \lambda_0^2) p_i = 0$, $i = 1, 2$. Solutions of these equations are represented in the form

$$p_1(x) = a_1\varphi_1(x) + a_2\varphi_2(x), \quad p_2(x) = a_3\varphi_1(x) + a_4\varphi_2(x), \tag{8}$$

where

$$\lambda_0^2 = -\frac{k(m_1 + m_2)}{m_1 m_2}, \quad a_1 = a_3 = \frac{2}{m_1 + m_2}, \quad a_2 = -\frac{m_1 - m_2}{m_1(m_1 + m_2)},$$

$$a_4 = -\frac{m_1 - m_2}{m_2(m_1 + m_2)}; \quad \triangle\varphi_1 = 0, \quad (\triangle + \lambda_0^2)\varphi_2 = 0,$$

Taking into account (8), we write

$$\beta_1 p_1 + \beta_2 p_2 = a\varphi_2 + b\varphi_1, \tag{9}$$

where

$$a = (\beta_1 + \beta_2)a_1, \quad b = \beta_1 a_2 + \beta_2 a_4. \tag{10}$$

Problem B$_1$. The functions φ_1 and φ_2 in formulas (8) are unknown. From the conditions (2), for problem B_1 we can write

$$\varphi_1(z) = \frac{d_1(z)}{d} \equiv \Omega_1(z), \quad \varphi_2(z) = \frac{d_2(z)}{d} \equiv \Omega_2(z), \quad z \in S, \tag{11}$$

where

$$d=a_1a_4-a_2^2,\quad d_1(z)=a_4f_3(z)-a_2f_4(z),\quad d_2(z)=a_1f_4(z)-a_2f_3(z).$$

Taking into account (11), for the harmonic function $\varphi_1(x)$ we have:

$$\varphi_1(x)=\sum_{m=0}^{\infty}\left(\frac{R}{r}\right)^m(A_m\cos m\psi+B_m\sin m\psi), \tag{12}$$

where

$$r^2=x_1^2+x_2^2,\ \ x=(x_1,x_2)=(r,\psi),\quad A_0=\frac{1}{2\pi}\int\limits_0^{2\pi}\Omega_1(\theta)d\theta,$$

$$A_m=\frac{1}{\pi}\int\limits_0^{2\pi}\Omega_1(\theta)\cos m\theta d\theta,\quad B_m=\frac{1}{\pi}\int\limits_0^{2\pi}\Omega_1(\theta)\sin m\theta d\theta.$$

Taking into account (8), the values in the plane of metaharmonic function $\varphi_2(x)$ can be represented as follows [3]:

$$\varphi_2(x)=K_0(\lambda_0 r)C_0+\sum_{m=1}^{\infty}K_m(\lambda_0 r)(C_m\cos m\psi+D_m\sin m\psi), \tag{13}$$

where $K_m(\lambda_0 r)$ is the modified Hancel ' s function of an imaginary argument,

$$C_m=\frac{1}{\pi}\int_0^{2\pi}\Omega_2(\theta)\cos m\theta d\theta,\ \ D_m=\frac{1}{\pi}\int_0^{2\pi}\Omega_2(\theta)\sin m\theta d\theta,\ \ m=0,1,\ldots. \tag{14}$$

Using now formulas (8), with regard to (12) and (13), we can find values of the functions $p_1(x)$ and $p_2(x)$.

Problem B$_2$. Taking into account formulas (8), the boundary conditions of problem B_2 can be rewritten as

$$\partial_R\varphi_1(z)=F_1(z),\quad \partial_R\varphi_2(z)=F_2(z),\quad z\in S, \tag{15}$$

where $F_1(z)=\dfrac{1}{d}[a_4f_3(z)-a_2f_4(z)]$, $\quad F_2(z)=\dfrac{1}{d}[a_1f_4(z)-a_2f_3(z)]$, $\quad \partial_R\equiv\partial_n$.

Then the harmonic function $\varphi_1(x)$ can be represented in the form of a series:

$$\varphi_1(x)=c_0-\sum_{m=1}^{\infty}\frac{R}{m}\left(\frac{R}{r}\right)^m(A_m\cos m\psi+B_m\sin m\psi), \tag{16}$$

where c_0 is an arbitrary constant, $A_m=\dfrac{1}{\pi}\int\limits_0^{2\pi}F_1(\theta)\cos m\theta d\theta$ and $B_m=\dfrac{1}{\pi}\int\limits_0^{2\pi}F_1(\theta)\sin m\varphi d\theta$.

Expanding the function $F_2(z)$ into Fourier series and substituting (13) into (15), we obtain the representation of the metaharmonic function $\varphi_2(x)$ in the plane in the form

$$\varphi_2(x) = \frac{1}{\lambda_0} \sum_{m=1}^{\infty} \frac{K_m(\lambda_0 r)}{K'_m(\lambda_0 R)} \, (\alpha_m \cos m\psi + \beta_m \sin m\psi), \tag{17}$$

where α_m and β_m are the Fourier coefficients of the function $F_2(z)$,

$$K'_m(\zeta) = \partial_\zeta K_m(\zeta), \quad \partial_r K_m(\lambda_0 r) = \lambda_0 K'_m(\lambda_0 r).$$

Problem $\mathbf{A}_1$. A solution of equation $(1)_1$ is sought in the form of a sum

$$u(x) = v_0(x) + v(x), \tag{18}$$

where v_0 is a particular solution of equation $(1)_1$, and v is a general solution of the corresponding homogeneous equation $(1)_1$. Direct checking shows that v_0 has the form

$$v_0(x) = \frac{1}{\lambda + 2\mu} grad\Big[-\frac{a}{\lambda_0^2}\varphi_2(x) + b\varphi_0(x)\Big], \tag{19}$$

where a and b are defined by formulas (10), and φ_0 is a biharmonic function: $\triangle\varphi_0 = \varphi_1$.

A solution $v(x) = (v_1, v_2)$ of the homogeneous equation corresponding to $(1)_1$ is sought in the form

$$v_1(x) = \partial_1[\Phi_1(x) + \Phi_2(x)] - \partial_2\Phi_3(x), \quad v_2(x) = \partial_2[\Phi_1(x) + \Phi_2(x)] + \partial_1\Phi_3(x), \tag{20}$$

where

$$\begin{aligned} &\triangle\Phi_1(x) = 0, \quad \triangle\triangle\Phi_2(x) = 0, \quad \triangle\triangle\Phi_3(x) = 0, \\ &(\lambda + 2\mu)\partial_1\triangle\Phi_2(x) - \mu\partial_2\triangle\Phi_3(x) = 0, \\ &(\lambda + 2\mu)\partial_2\triangle\Phi_2(x) + \mu\partial_1\triangle\Phi_3(x) = 0, \end{aligned} \tag{21}$$

Φ_1, Φ_2, Φ_3 are the scalar functions.

Taking into account (18) and relying on the condition $(2)_I$, we can write

$$v(z) = \Psi(z), \tag{22}$$

where $\Psi(z) = f(z) - v_0(z)$ is the known vector; v_0 is defined by formula (19), and φ_1 and φ_2 by equalities (11). The value of the function φ_0 is defined by means of the equation $\triangle\varphi_0 = \varphi_1$; it has the form

$$\varphi_0(x) = \frac{R^2}{4} \sum_{m=2}^{\infty} \frac{1}{1-m} \Big(\frac{R}{r}\Big)^{m-2} (A_m \cos m\psi + B_m \sin m\psi) + \frac{A_0}{4} r^2, \tag{23}$$

where A_m and B_m are defined in (12).

In view of (21), we can represent the harmonic function Φ_1 and biharmonic functions Φ_2 and Φ_3 in the form

$$\begin{aligned}
\Phi_1(x) &= \sum_{m=0}^{\infty}\left(\frac{R}{r}\right)^{m}(X_{m_1}\cdot\nu_m(\psi)),\\
\Phi_2(x) &= R^2\sum_{m=0}^{\infty}\left(\frac{R}{r}\right)^{m-2}(X_{m_2}\cdot\nu_m(\psi)),\\
\Phi_3(x) &= R^2\frac{\lambda+2\mu}{\mu}\sum_{m=0}^{\infty}\left(\frac{R}{r}\right)^{m-2}(X_{m_2}\cdot s_m(\psi)),
\end{aligned}\tag{24}$$

where X_{mk} are the unknown two-component vectors, $k=1,2$;

$$\nu_m(\psi)=(\cos m\psi,\sin m\psi),\quad s_m(\psi)=(-\sin m\psi,\cos m\psi),\quad x=(r,\psi), x\in D.$$

Substituting (24) into (20), the condition (22) for every m results in a system of linear algebraic equations whose solution is written as follows:

$X_{01}=\dfrac{\alpha_0 R}{4},\quad X_{02}=\dfrac{\beta_0 R}{4},$

$X_{m1}=\dfrac{R(\alpha_m+\beta_m)}{2m(\lambda+3\mu)}[2\mu+(\lambda+\mu)m]-\dfrac{R\alpha_m}{m},\quad X_{m2}=\dfrac{\mu(\alpha_m+\beta_m)}{2(\lambda+3\mu)R},$

$m=1,2,...$; α_m and β_m are the Fourier coefficients of, respectively, the normal and tangential components of the function $\Psi(z)=f(z)-v_0(z)$, $z\in S$.

Thus the solution of problem A_1 is represented by the sum (21) in which $v(x)$ is defined by means of formula (23), and $v_0(x)$ by formula (22).

Problem $\mathbf{A}_2$. Taking into account (3) and (9), the boundary condition $(2)_{II}$ can be rewritten as

$$T(\partial_z,n)v(z)=\Psi(z),\quad z\in S,\tag{25}$$

where

$$\Psi(z)=f(z)+n(z)[a\varphi_2(z)+b\varphi_1(z)]-T(\partial_z,n)v_0(z)$$

is the known vector, $\Psi=(\Psi_1,\Psi_2)$.

We substitute (24) first into (23) and then into (25). For the unknowns X_{m1} and X_{m2} we obtain a system of algebraic equations:

$$2(\lambda+2\mu)X_{01}=\frac{\alpha_0}{2},\quad 2(\lambda+2\mu)=\frac{\beta_0}{2},$$

$$m[\lambda+2\mu(m+1)]X_{m1}+\{(\lambda+2\mu)(1-m)(2-m+\frac{\lambda+2\mu}{\mu}m)$$
$$-\lambda mR^2[m+\frac{\lambda+2\mu}{\mu}(2-m)]\}X_{m2}=\alpha_m R^2,$$

$$-m(1+2\mu)X_{m1}+R^2[m(3-2m)+\frac{\lambda+2\mu}{\mu}(m^2-3m+2)]X_{m2}=\beta_m\frac{R^2}{\mu},$$

$m=1,2,...$; α_m and β_m are the Fourier coefficients of, respectively, the normal and tangential components of the function $\Psi(z)=f(z)+n(z)[a\varphi_2(z)+b\varphi_1(z)]-T(\partial_z,n)v_0(z)$;

v_0 is defined by means of formula (19) in which $\varphi_0(x)$ for problem B_1 has the form (23) and for problem B_2 the form

$$\varphi_0(x) = \frac{R^3}{4} \sum_{m=2}^{\infty} \frac{1}{m(1-m)} \Big(\frac{R}{r}\Big)^{m-2} (A_m \cos m\psi + B_m \sin m\psi),$$

where A_m and B_m are defined in (16).

Conditions: $f, p_1, p_2 \in C^3(S)$ - in problem A_1 and conditions: $f, p_1, p_2 \in C^2(S)$ in problem A_2, ensure absolutely and uniformly convergence of series obtained for $v(x)$ and $v_0(x)$ and also, (8).

Having solved problems A_1, A_2, B_1 and B_2, we can write solutions of the initial problems I and II.

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