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ON EFFECTS OF CONSTANT DELAY PERTURBATION AND THE DISCONTINUOUS INITIAL CONDITION IN VARIATION FORMULAS OF SOLUTION OF DELAY CONTROLLED FUNCTIONAL-DIFFERENTIAL EQUATION

Tadumadze T., Gorgodze N.

Abstract. Variation formulas of solution (variation formulas) are proved for a controlled non-linear delay functional-differential equation with the discontinuous initial condition, under perturbations of initial moment, delay parameter, initial vector, initial and control functions. The effects of delay perturbation and the discontinuous initial condition are discovered in the variation formulas. The discontinuity of the initial condition means that the values of the initial function and the trajectory, generally, do not coincide at the initial moment.

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1. Introduction

Linear representation of the main part of the increment of a solution of an equation with respect to perturbations is called the variation formula. The variation formula allows one to construct an approximate solution of the perturbed equation in an analytical form on the one hand, and in the theory of optimal control plays the basic role in proving the necessary conditions of optimality [1-11], on the other. Variation formulas for various classes of functional-differential equations without perturbation of delay are given in [6,10,12-14].Variation formulas for delay functional-differential equations with the continuous and discontinuous initial condition taking into consideration constant delay perturbation are proved in [15] and [16], respectively. Variation formulas for controlled delay functional-differential equations with the continuous initial condition taking into consideration constant delay perturbation are proved in [17]. In this paper the variation formulas are proved for the controlled delay functional-differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau_0), u_0(t))$$

with the discontinuous initial condition

$$x(t) = \varphi_0(t), t \in [t_{00} - \tau_0, t_{00}), x(t_{00}) = x_{00}$$

under perturbations of initial moment t_{00}, delay parameter τ_0, initial vector x_{00}, initial function $\varphi_0(t)$ and control function $u_0(t)$.

2. Notation and auxiliary assertions

Let R^n_x be the n-dimensional vector space of points $x = (x^1, ..., x^n)^T$, where T means transpose; suppose that $O \subset R^n_x$ and $V \subset R^r_u$ are open sets. Let the n-dimensional function $f(t,x,y,u)$ satisfy the following conditions: for almost all $t \in I = [a,b]$, the function $f(t,\cdot) : O^2 \times V \to R_x^n$ is continuously differentiable; for any $(x,y,u) \in O^2 \times V$, the functions $f(t,x,y,u), f_x(\cdot), f_y(\cdot), f_u(\cdot)$ are measurable on I; for arbitrary compacts $K \subset O, U \subset V$ there exists a function $m_{K,U}(\cdot) \in L(I,[0,\infty))$, such that for any $(x,y,u) \in K^2 \times U$ and for almost all $t \in I$ the following inequality is fulfilled

$$| f(t,x,y,u) | + | f_x(\cdot) | + | f_y(\cdot) | + | f_u(\cdot) | \leq m_{K,U}(t).$$

Further, let $0 < \tau_1 < \tau_2$ be given numbers; Let E_φ be the space of continuous functions $\varphi : I_1 \to R_x^n$, where $I_1 = [\hat{\tau}, b], \hat{\tau} = a - \tau_2; \Phi = \{\varphi \in E_\varphi : \varphi(t) \in O, t \in I_1\}$ is a set of initial functions; let E_u be the space of bounded measurable functions $u : I \to R_u^r$ and let $\Omega = \{u \in E_u : clu(I) \subset V\}$ be a set of control functions, where $u(I) = \{u(t) : t \in I\}$ and $clu(I)$ is the closer of the set $u(I)$.

To each element $\mu = (t_0, \tau, x_0, \varphi, u) \in \Lambda = (a,b) \times (\tau_1, \tau_2) \times O \times \Phi \times \Omega$ we assign the controlled delay functional-differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau), u(t)) \tag{2.1}$$

with the initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0), x(t_0) = x_0. \tag{2.2}$$

The condition (2.2) is said to be the discontinuous initial condition since generally $x(t_0) \neq \varphi(t_0)$.

Definition 2.1. Let $\mu = (t_0, \tau, x_0, \varphi, u) \in \Lambda$. A function $x(t) = x(t;\mu) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b)$, is called a solution of equation (2.1) with the initial condition (2.2) or a solution corresponding to μ and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (2.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (2.1) almost everywhere on $[t_0, t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, u_0) \in \Lambda$ be a fixed element. In the space $E_\mu = R_{t_0}^1 \times R_\tau^1 \times R_x^n \times E_\varphi \times E_u$ we introduce the set of variations:

$$V = \{\delta\mu = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta u) \in E_\mu - \mu_0 : | \delta t_0 | \leq \alpha, | \delta\tau | \leq \alpha, | \delta x_0 | \leq \alpha,$$

$$\delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \delta u = \sum_{i=1}^{k} \lambda_i \delta u_i, | \lambda_i | \leq \alpha, i = \overline{1,k}\}, \tag{2.3}$$

where $\delta\varphi_i \in E_\varphi - \varphi_0, \delta u_i \in E_u - u_0, i = \overline{1,k}$ are fixed functions ; $\alpha > 0$ is a fixed number.

Lemma 2.1. *Let $x_0(t)$ be the solution corresponding to $\mu_0 = (t_{00}, \tau_0, x_0, \varphi_0, u_0) \in \Lambda$ and defined on $[\hat{\tau}, t_{10}], t_{10} \in (t_{00}, b)$ and let $K_0 \subset O$ and $U_0 \subset V$ be compact sets containing neighborhoods of sets $\varphi_0(I_1) \cup x_0([t_{00}, t_{10,}])$ and $clu_0(I)$, respectively. Then there exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that, for any $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$. In addition, a solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to this element. Moreover,*

$$\begin{cases} x(t; \mu_0 + \varepsilon\delta\mu) \in K_0, t \in [\hat{\tau}, t_{10} + \delta_1], \\ u_0(t) + \varepsilon\delta u(t) \in U_0, t \in I. \end{cases} \tag{2.4}$$

This lemma is a result of Theorem 5.3 in [18, p.111].

Remark 2.1. Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\hat{\tau}, t_{10}+\delta_1]$. Therefore, in the sequel the solution $x_0(t)$ is assumed to be defined on the interval $[\hat{\tau}, t_{10}+\delta_1]$.

Lemma 2.1 allows one to define the increment of the solution $x_0(t)=x(t;\mu_0)$:

$$\begin{cases}\Delta x(t)=\Delta x(t;\varepsilon\delta\mu)=x(t;\mu_0+\varepsilon\delta\mu)-x_0(t),\\ (t,\varepsilon,\delta\mu)\in[\hat{\tau},t_{10}+\delta_1]\times[0,\varepsilon_1]\times V.\end{cases} \tag{2.5}$$

Lemma 2.2. *Let the following conditions hold:*

2.1. $t_{00}+\tau_0<t_{10}$;

2.2. *the function* $\varphi_0(t), t\in I_1$ *is absolutely continuous and the function* $\dot{\varphi}_0(t)$ *is bounded;*

2.3. *there exist compact sets* $K_0\subset O$ *and* $U_0\subset V$ *containing neighborhoods of sets* $\varphi_0(J_1)\cup x_0([t_{00},t_{10}])$ *and* $clu_0(I)$, *respectively, such that the function* $f(t,x,y,u)$ *is bounded on the set* $I\times K_0^2\times U_0$;

2.4. *there exists the limit*

$$\lim_{w\to w_0} f(w,u_0(t))=f_0^-, w=(t,x,y)\in(a,t_{00}]\times O^2,$$

where $w_0=(t_{00},x_{00},\varphi_0(t_{00}-\tau_0))$. *Then there exist numbers* $\varepsilon_2\in(0,\varepsilon_1]$ *and* $\delta_2\in(0,\delta_1]$ *such that*

$$\max_{t\in[t_{00},t_{10}+\delta_2]} |\Delta x(t)|\le O(\varepsilon\delta\mu)^3 \tag{2.6}$$

for arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V^-$, *where* $V^-=\{\delta\mu\in V:\delta t_0\le 0,\delta\tau\le 0\}$. *Moreover,*

$$\Delta x(t_{00})=\varepsilon\Big[\delta x_0-f_0^-\delta t_0\Big]+o(\varepsilon\delta\mu). \tag{2.7}$$

Lemma 2.3. *Let the conditions 2.1-2.3 of Lemma 2.2 hold, and there exists the limit*

$$\lim_{w\to w_0} f(w,u_0(t))=f_0^+, w=(t,x,y)\in[t_{00},b)\times O^2.$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1]$ *and* $\delta_2\in(0,\delta_1]$ *such that the inequality*

$$\max_{t\in[t_0,t_{10}+\delta_2]} |\Delta x(t)|\le O(\varepsilon\delta\mu), \tag{2.8}$$

is valid for arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V^+$, *where* $t_0=t_{00}+\varepsilon\delta t_0, V^+=\{\delta\mu\in V:\delta t_0\ge 0,\delta\tau\ge 0\}$. *Moreover,*

$$\Delta x(t_0)=\varepsilon\Big[\delta x_0-f_0^+\delta t_0\Big]+o(\varepsilon\delta\mu). \tag{2.9}$$

Lemmas 2.2 and 2.3 can be proved in analogy to Lemma 2.3 (see [15]).

[3] Here and throughout the following, the symbols $O(t;\varepsilon\delta\mu), o(t;\varepsilon\delta\mu)$ stand for quantities (scalar or vector) that have the corresponding order of smallness with respect to ε uniformly with respect to t and $\delta\mu$.

Lemma 2.4. *Let the conditions of Lemma 2.2 hold. Then*

$$\alpha(t_{00}+\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu)=\int_{t_{00}+\tau_0}^{t_{10}+\delta_2}\zeta(t)\Big[|\Delta x(t-\tau)-\Delta x(t-\tau_0)|\Big]dt$$
$$\le o(\varepsilon\delta\mu), \tag{2.10}$$

for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_2]\times V^-$, *where* $\tau=\tau_0+\varepsilon\delta\tau,\zeta(\cdot)\in L(J,[0,\infty))$, *about* ε_2 *and* δ_2 *see Lemma 2.2.*

Proof. It is obvious that $t-\tau\ge t_{00}$ and $t-\tau_0\ge t_{00}$ for $t\in[t_{00}+\tau_0,t_{10}+\delta_2]$. Therefore,

$$\alpha(t_{00}+\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu)\le\int_{t_{00}+\tau_0}^{t_{10}+\delta_2}\zeta(t)\Big[\int_{t-\tau_0}^{t-\tau}|\dot{\Delta}x(\xi)|d\xi\Big]dt$$
$$=\int_{t_{00}+\tau_0}^{t_{10}+\delta_2}\zeta(t)\Big[\int_{t-\tau_0}^{t-\tau}\theta(\xi;\varepsilon\delta\mu)d\xi\Big]dt,$$

where

$$\theta(\xi;\varepsilon\delta\mu)=|f(\xi,x_0(\xi)+\Delta x(\xi),x_0(\xi-\tau)+\Delta x(\xi-\tau),u_0(\xi)+\varepsilon\delta u(\xi))$$
$$-f[\xi]|,f[\xi]=f(\xi,x_0(\xi),x_0(\xi-\tau_0),u_0(\xi))$$

see (2.5).

a) Let $t_{00}+2\tau_0\le t_{10}$ and $\varepsilon_2\in(0,\varepsilon_1]$ be so small that $t_0+2\tau>t_{00}+\tau_0,\forall(\varepsilon,\delta\mu)\in(0,\varepsilon_2]\times V^-$, then we have

$$\alpha(t_{00}+\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu)=\alpha(t_{00}+\tau_0,t_0+2\tau;\varepsilon\delta\mu)+\alpha(t_0+2\tau,t_{00}+2\tau_0;\varepsilon\delta\mu)$$
$$+\alpha(t_{00}+2\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu).$$

The function $\theta(\xi;\varepsilon\delta\mu)$ is bounded (see the condition 2.3 of Lemma 2.2), therefore

$$\alpha(t_0+2\tau,t_{00}+2\tau_0;\varepsilon\delta\mu)\le o(\varepsilon\delta\mu).$$

We note that there exists $L(\cdot)\in L(I,[0,\infty))$ such that

$$|f(t,x_1,y_1,u_1)-f(t,x_2,y_2,u_2)|\le L(t)\Big(|x_1-x_2|+|y_1-y_2|+|u_1-u_2|\Big),$$
$$t\in I,(x_i,y_i,u_i)\in K_0^2\times U_0,i=1,2,3.$$

It is not difficult to see that

$$\alpha(t_{00}+\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu)\le\alpha_1(t_{00}+\tau_0,t_0+2\tau;\varepsilon\delta\mu)+o(\varepsilon\delta\mu)$$
$$+\alpha_1(t_{00}+2\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu), \tag{2.11}$$

where

$$\alpha_1(t',t'';\varepsilon\delta\mu)=\int_{t'}^{t''}\zeta(t)\alpha_2(t;\varepsilon\delta\mu)dt,\alpha_2(t;\varepsilon\delta\mu)$$

$$= \int_{t-\tau_0}^{t-\tau} L(\xi)\Big\{|\Delta x(\xi)| + |x_0(\xi-\tau) - x_0(\xi-\tau_0)| + |\Delta x(\xi-\tau)| + \varepsilon|\delta u(\xi)|\Big\}d\xi.$$

If $t \in [t_{00}+\tau_0, t_0+2\tau]$ and $\xi \in [t-\tau_0, t-\tau]$ then $\xi \geq t_{00}, \xi-\tau \leq t_0, \xi-\tau_0 \leq t_0$. Therefore,

$$|\Delta x(\xi)| \leq O(\varepsilon\delta\mu), |x_0(\xi-\tau) - x_0(\xi-\tau_0)| = |\varphi_0(\xi-\tau) - \varphi_0(\xi-\tau_0)|$$

$$= \int_{t-\tau_0}^{t-\tau} |\dot{\varphi}_0(\xi)|d\xi = O(\varepsilon\delta\mu), |\Delta x(\xi-\tau)| = \varepsilon|\delta\varphi(\xi-\tau)|. \tag{2.12}$$

Thus,

$$\alpha_1(t_{00}+\tau_0, t_0+2\tau; \varepsilon\delta\mu) \leq o(\varepsilon\delta\mu). \tag{2.13}$$

Further, if $t \in [t_{00}+2\tau_0, t_{10}+\delta_2]$ and $\xi \in [t-\tau_0, t-\tau]$ then $\xi \geq t_{00}+\tau_0, \xi-\tau \geq t_{00}, \xi-\tau_0 \geq t_{00}$. Therefore,

$$|\Delta x(\xi)| \leq O(\varepsilon\delta\mu), |x_0(\xi-\tau) - x_0(\xi-\tau_0)| = \int_{t-\tau_0}^{t-\tau} |\dot{x}_0(\xi)|d\xi$$

$$= \int_{t-\tau_0}^{t-\tau} |f[\xi]|d\xi = O(\varepsilon\delta\mu), |\Delta x(\xi-\tau)| = O(\varepsilon\delta\mu).$$

Consequently,

$$\alpha_1(t_{00}+2\tau_0, t_{10}+\delta_2; \varepsilon\delta\mu) \leq o(\varepsilon\delta\mu). \tag{2.14}$$

From (2.11) by virtue (2.13) and (2.14) we obtain (2.10).

b) Let $t_{00}+2\tau_0 > t_{10}$ and, ε_2 and δ_2 be so small that $t_{00}+2\tau > t_{10}+\delta_2$. It is clear that

$$\alpha(t_{00}+\tau_0, t_{10}+\delta_2; \varepsilon\delta\mu) \leq \alpha_1(t_{00}+\tau_0, t_{10}+\delta_2; \varepsilon\delta\mu).$$

If $t \in [t_{00}+\tau_0, t_{10}+\delta_2]$ and $\xi \in [t-\tau_0, t-\tau]$ then $\xi \geq t_{00}, \xi-\tau \leq t_0, \xi-\tau_0 \leq t_0$. Therefore,

$$\alpha_1(t_{00}+\tau_0, t_{10}+\delta_2; \varepsilon\delta\mu) \leq o(\varepsilon\delta\mu)$$

(see (2.12)). Lemma 2.4 is proved.

Lemma 2.5. *Let the conditions of Lemma 2.3 hold. Then*

$$\int_{t_0+\tau}^{t_{10}+\delta_2} \zeta(t)\Big[|\Delta x(t-\tau) - \Delta x(t-\tau_0)|\Big]dt \leq o(\varepsilon\delta\mu).$$

for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2] \times V^+$.

This Lemma can be proved in analogy to Lemma 2.4.

3. Formulation of main results

Theorem 3.1. *Let the conditions of Lemma 2.2 hold.Moreover, there exits the limit*

$$\lim_{(w_1,w_2)\to(w_{01},w_{02})} [f(w_1, u_0(t)) - f(w_2, u_0(t))] = f_1^-, w_i \in (a, t_{00}+\tau_0] \times O^2, i = 1, 2,$$

where

$$w_{01} = (t_{00} + \tau_0, x_0(t_{00} + \tau_0), x_{00}), w_{02} = (t_{00} + \tau_0, x_0(t_{00} + \tau_0), \varphi_0(t_{00})).$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1]$ *and* $\delta_2 \in (0, \delta_1]$ *such that*

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu) \tag{3.1}$$

for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V^-$ *and*

$$\delta x(t; \delta\mu) = -\Big\{Y(t_{00}; t)f_0^- + Y(t_{00} + \tau_0; t)f_1^-\Big\}\delta t_0$$
$$-Y(t_{00} + \tau_0; t)f_1^-\delta\tau + \beta(t; \delta\mu), \tag{3.2}$$

where

$$\beta(t; \delta\mu) = Y(t_{00}; t)\delta x_0 + \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi + \tau_0; t)f_y[\xi + \tau_0]\delta\varphi(\xi)d\xi$$
$$-\Big\{\int_{t_{00}}^{t} Y(\xi; t)f_y[\xi]\dot{x}_0(\xi - \tau_0)d\xi\Big\}\delta\tau + \int_{t_{00}}^{t} Y(\xi; t)f_u[\xi]\delta u(\xi)d\xi. \tag{3.3}$$

Here $Y(\xi; t)$ *is the* $n \times n$*-matrix function satisfying the linear functional-differential equation with advanced argument*

$$Y_\xi(\xi; t) = -Y(\xi; t)f_x[\xi] - Y(\xi + \tau_0; t)f_y[\xi + \tau_0], \xi \in [t_{00}, t], \tag{3.4}$$

and the condition

$$Y(\xi; t) = \begin{cases} H \ for \ \xi = t, \\ \Theta \ for \ \xi > t, \end{cases} \tag{3.5}$$

$$f_x = \frac{\partial}{\partial x}f, f_x[\xi] = f_x(\xi, x_0(\xi), x_0(\xi - \tau_0), u_0(\xi));$$

H *is the identity matrix and* Θ *is the zero matrix.*

Some comments. The expression (3.2) is called the variation formula.

c1. Theorem 3.1 corresponds to the case when the variations at the points t_{00} and τ_0 are performed simultaneously on the left.

c2. The summand

$$-\Big\{Y(t_{00} + \tau_0; t)f_1^- + \int_{t_{00}}^{t} Y(\xi; t)f_y[\xi]\dot{x}_0(\xi - \tau_0)d\xi\Big\}\delta\tau$$

in formula (3.2) (see also (3.3)) is the effect of perturbation of the delay τ_0.

c3. The expression

$$-\Big\{Y(t_{00}; t)f_0^- + Y(t_{00} + \tau_0; t)f_1^-\Big\}\delta t_0$$

is the effect of discontinuous initial condition (2.2) and perturbation of the initial moment t_{00}.

c4. The expression

$$Y(t_{00}; t)\delta x_0 + \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi + \tau_0; t)f_y[\xi + \tau_0]\delta\varphi(\xi)d\xi + \int_{t_{00}}^{t} Y(\xi; t)f_u[\xi]\delta u(\xi)d\xi$$

in formula (3.3) is the effect of perturbations of the initial vector x_0, initial $\varphi_0(t)$ and control $u_0(t)$ functions.
c5. The variation formula allows one to obtain an approximate solution of the perturbed functional-differential equation

$$\dot{x}(t) = f(t, x(t), x(t - \tau_0 - \varepsilon\delta\tau), u_0(t) + \varepsilon\delta u(t))$$

with the perturbed initial condition

$$x(t) = \varphi_0(t) + \varepsilon\delta\varphi(t), t \in [\hat{\tau}, t_{00} + \varepsilon\delta t_0), x(t_{00}) = x_{00} + \varepsilon\delta x_0.$$

In fact, for a sufficiently small $\varepsilon \in (0, \varepsilon_2]$ from (3.1) it follows

$$x(t; \mu_0 + \varepsilon\delta\mu) \approx x_0(t) + \varepsilon\delta x(t; \delta\mu)$$

(see (2.5)).
c6. Finally we note that the variation formula which is proved in the present work doesn't follows from the formula proved in [15].

Theorem 3.2. *Let the conditions of Lemma 2.3 hold.Moreover, there exits the limit*

$$\lim_{(w_1,w_2)\to(w_{01},w_{02})} [f(w_1, u_0(t)) - f(w_2, u_0(t))] = f_1^+, w_i \in [t_{00} + \tau_0, b) \times O^2, i = 1, 2.$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1]$ and $\delta_2 \in (0, \delta_1]$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V^+$, formula (3.1) holds and

$$\delta x(t; \delta\mu) = -\Big\{Y(t_{00}; t)f_0^+ + Y(t_{00} + \tau_0; t)f_1^+\Big\}\delta t_0$$
$$-Y(t_{00} + \tau_0; t)f_1^+\delta\tau + \beta(t; \delta\mu). \quad (3.6)$$

Theorem 3.2 corresponds to the case when the variations at the points t_{00} and τ_0 are performed simultaneously on the right. Theorems 3.1 and 3.2 are proved by a scheme given in [10].

4. Proof of Theorem 3.1

Here and in what follows we shall assume that $t_0 = t_{00} + \varepsilon\delta t_0, \tau = \tau_0 + \varepsilon\delta\tau, \varphi(t) = \varphi_0(t) + \varepsilon\delta\varphi(t), u(t) = u_0(t) + \varepsilon\delta u(t)$. Let $\varepsilon_2 \in (0, \varepsilon_1]$ be so small (see Lemma 2.2) that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2] \times V^-$ the following inequalities hold

$$t_{00} - \tau \le t_0, t_0 + \tau \ge t_{00}.$$

The function $\Delta x(t)$ (see (2.5)) satisfies the equation

$$\dot{\Delta}x(t) = f(t, x_0(t) + \Delta x(t), x_0(t - \tau) + \Delta x(t - \tau), u(t)) - f[t]$$
$$= f_x[t]\Delta x(t) + f_y[t]\Delta x(t - \tau_0) + \varepsilon f_u[t]\delta u(t) + r(t; \varepsilon\delta\mu) \quad (4.1)$$

on the interval $[t_{00}, t_{10}+\delta_2]$, where

$$r(t;\varepsilon\delta\mu) = f(t, x_0(t)+\Delta x(t), x_0(t-\tau)+\Delta x(t-\tau), u(t)) - f[t]$$
$$-f_x[t]\Delta x(t) - f_y[t]\Delta x(t-\tau_0) - \varepsilon f_u[t]\delta u(t), \tag{4.2}$$

By using the Cauchy formula ([10], p.21), one can represent the solution of equation (4.1) in the form

$$\Delta x(t) = Y(t_{00};t)\Delta x(t_{00}) + \varepsilon\int_{t_{00}}^{t} Y(\xi;t) f_u[\xi]\delta u(\xi)d\xi$$
$$+\sum_{i=0}^{1} R_i(t;t_{00},\varepsilon\delta\mu), t\in[t_{00}, t_{10}+\delta_2], \tag{4.3}$$

where

$$\begin{cases} R_0(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t) f_y[\xi+\tau_0]\Delta x(\xi)d\xi, \\ R_1(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}}^{t} Y(\xi;t) r(\xi;\varepsilon\delta\mu)d\xi \end{cases} \tag{4.4}$$

and $Y(\xi;t)$ is the matrix function satisfying equation (3.4) and condition (3.5).

Let a number $\delta_2 \in (0,\delta_1]$ be so small that $t_{00}+\tau_0 < t_{10}-\delta_2$. The function $Y(\xi;t)$ is continuous on the set

$$\Pi = \{(\xi,t):\ \xi\in[t_{00}, t_{00}+\tau_0], t\in[t_{10}-\delta_2, t_{10}+\delta_2]\}$$

([10], Lemma 2.1.7). Therefore,

$$Y(t_{00};t)\Delta x(t_{00}) = \varepsilon Y(t_{00};t)\Big[\delta x_0 - f_0^{-}\delta t_0\Big] + o(t;\varepsilon\delta\mu) \tag{4.5}$$

(see (2.7)). One can readily see that

$$R_0(t;t_{00},\varepsilon\delta\mu) = \varepsilon\int_{t_{00}-\tau_0}^{t_0} Y(\xi+\tau_0;t) f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$
$$+\int_{t_0}^{t_{00}} Y(\xi+\tau_0;t) f_y[\xi+\tau_0]\Delta x(\xi)d\xi = \varepsilon\int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t) f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$
$$+\int_{t_0+\tau_0}^{t_{00}+\tau_0} Y(\xi;t) f_y[\xi]\Delta x(\xi-\tau_0)d\xi + o(t;\varepsilon\delta\mu), \tag{4.6}$$

where

$$o(t;\varepsilon\delta\mu) = -\varepsilon\int_{t_0}^{t_{00}} Y(\xi+\tau_0;t) f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi.$$

For $t\in[t_{10}-\delta_2, t_{10}+\delta_2]$ we have

$$R_1(t;t_{00},\varepsilon\delta\mu) = \sum_{i=1}^{3}\alpha_i(t;\varepsilon\delta\mu) \tag{4.7}$$

$$\alpha_1(t;\varepsilon\delta\mu)=\int_{t_{00}}^{t_0+\tau} r_1(\xi;t,\varepsilon\delta\mu)d\xi, \alpha_2(t;\varepsilon\delta\mu)=\int_{t_0+\tau}^{t_{00}+\tau_0} r_1(\xi;t,\varepsilon\delta\mu)d\xi,$$

$$\alpha_3(t;\varepsilon\delta\mu)=\int_{t_{00}+\tau_0}^{t} r_1(\xi;t,\varepsilon\delta\mu)d\xi, r_1(\xi;t,\varepsilon\delta\mu)=Y(\xi;t)r(\xi;\varepsilon\delta\mu).$$

We introduce the notations:

$$f[t;s,\varepsilon\delta\mu]=f(t,x_0(t)+s\Delta x(t),x_0(t-\tau_0)+s\{x_0(t-\tau)-x_0(t-\tau_0)$$

$$+\Delta x(t-\tau)\},u_0(t)+s\varepsilon\delta u(t)),\sigma(t;s,\varepsilon\delta\mu)=f_x[t;s,\varepsilon\delta\mu]-f_x[t],$$

$$\rho(t;s,\varepsilon\delta\mu)=f_y[t;s,\varepsilon\delta\mu]-f_y[t],\vartheta(t;s,\varepsilon\delta\mu)=f_u[t;s,\varepsilon\delta\mu]-f_u[t].$$

It is easy to see that

$$f(t,x_0(t)+\Delta x(t),x_0(t-\tau)+\Delta x(t-\tau),u_0(t)+\varepsilon\delta u(t))-f[t]$$

$$=\int_0^1\frac{d}{ds}f[t;s,\varepsilon\delta\mu]ds=\int_0^1\Big\{f_x[t;s,\varepsilon\delta\mu]\Delta x(t)+f_y[t;s,\varepsilon\delta\mu]\{x_0(t-\tau)$$

$$-x_0(t-\tau_0)+\Delta x(t-\tau)\}+\varepsilon f_u[t;s,\varepsilon\delta\mu]\delta u(t)\Big\}ds$$

$$=\Big[\int_0^1\sigma(t;s,\varepsilon\delta\mu)ds\Big]\Delta x(t)+\Big[\int_0^1\rho(t;s,\varepsilon\delta\mu)ds\Big]\{x_0(t-\tau)$$

$$-x_0(t-\tau_0)+\Delta x(t-\tau)\}+\varepsilon\Big[\int_0^1\vartheta(t;s,\varepsilon\delta\mu)ds\Big]\delta u(t)$$

$$+f_x[t]\Delta x(t)+f_y[t]\{x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau)\}+\varepsilon f_u[t]\delta u(t).$$

On account of the last relation we have

$$\alpha_1(t;\varepsilon\delta\mu)=\sum_{i=1}^{5}\alpha_{1i}(t;\varepsilon\delta\mu),$$

where

$$\alpha_{11}(t;\varepsilon\delta\mu)=\int_{t_{00}}^{t_0+\tau}Y(\xi;t)\sigma_1(\xi;\varepsilon\delta\mu)\Delta x(\xi)d\xi,$$

$$\sigma_1(\xi;\varepsilon\delta\mu)=\int_0^1\sigma(\xi;s,\varepsilon\delta\mu)ds,\ \alpha_{12}(t;\varepsilon\delta\mu)$$

$$=\int_{t_{00}}^{t_0+\tau}Y(\xi;t)\rho_1(\xi;\varepsilon\delta\mu)\{x_0(\xi-\tau)-x_0(\xi-\tau_0)+\Delta x(\xi-\tau)\}d\xi,$$

$$\rho_1(\xi;\varepsilon\delta\mu)=\int_0^1\rho(\xi;s,\varepsilon\delta\mu)ds,\alpha_{13}(t;\varepsilon\delta\mu)$$
$$=\varepsilon\int_{t_{00}}^{t_0+\tau}Y(\xi;t)\vartheta_1(\xi;\varepsilon\delta\mu)\delta u(\xi)d\xi,\ \vartheta_1(\xi;\varepsilon\delta\mu)$$
$$=\int_0^1\vartheta(\xi;s,\varepsilon\delta\mu)ds,\ \alpha_{14}(t;\varepsilon\delta\mu)=\int_{t_{00}}^{t_0+\tau}Y(\xi;t)f_y[\xi]\{\Delta x(\xi-\tau)$$
$$-\Delta x(\xi-\tau_0)\}d\xi,\ \alpha_{15}(t;\varepsilon\delta\mu)=\int_{t_{00}}^{t_0+\tau}Y(\xi;t)f_y[\xi]\{x_0(\xi-\tau)$$
$$-x_0(\xi-\tau_0)\}d\xi$$

For $\xi\in[t_{00},t_0+\tau]$ we have

$$\begin{cases}|\Delta x(\xi)|\le O(\varepsilon\delta\mu),\Delta x(\xi-\tau)=\varepsilon\delta\varphi(\xi-\tau),\\ \Delta x(\xi-\tau)-\Delta x(\xi-\tau_0)=\varepsilon[\delta\varphi(\xi-\tau)-\delta\varphi(\xi-\tau_0)]\\ x_0(\xi-\tau)-x_0(\xi-\tau_0)=\varphi_0(\xi-\tau)-\varphi_0(\xi-\tau_0)\end{cases}\tag{4.8}$$

(see (4.2)). The function $\varphi_0(t)$ is absolutely continuous, therefore for each fixed Lebesgue point $\xi\in(t_{00},t_{00}+\tau_0)$ of function $\dot\varphi_0(\xi-\tau_0)$ we get

$$\varphi_0(\xi-\tau)-\varphi_0(\xi-\tau_0)=\int_\xi^{\xi-\varepsilon\delta\tau}\dot\varphi_0(s-\tau_0)ds$$
$$=-\varepsilon\dot\varphi_0(\xi-\tau_0)\delta\tau+\gamma(\xi;\varepsilon\delta\mu),\tag{4.9}$$

with

$$\lim_{\varepsilon\to0}\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}=0\ \text{ uniformly for }\delta\mu\in V^-.\tag{4.10}$$

Thus, (4.9) and (4.10) are valid for almost all points of the interval $(t_{00},t_{00}+\tau_0)$. From (4.9) taking into account boundedness of the function $\dot\varphi_0(t)$ it follows

$$|\ \varphi_0(\xi-\tau)-\varphi_0(\xi-\tau_0)\ |\le O(\varepsilon\delta\mu)\text{ and }\left|\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}\right|\le const.\tag{4.11}$$

Consequently, for $\alpha_{1i}(t;\varepsilon\delta\mu)),i=\overline{1,4}$ we have

$$\begin{cases}|\ \alpha_{11}(t;\varepsilon\delta\mu)|\le\|\ Y\ \|\ O(\varepsilon\delta\mu)\sigma_2(\varepsilon\delta\mu),\\ |\ \alpha_{12}(t;\varepsilon\delta\mu)|\le\|\ Y\ \|\ O(\varepsilon\delta\mu)\rho_2(\varepsilon\delta\mu),\\ |\ \alpha_{13}(t;\varepsilon\delta\mu)|\le\varepsilon\ \|\ Y\ \|\ \vartheta_2(\varepsilon\delta\mu),\\ |\ \alpha_{14}(t;\varepsilon\delta\mu)|\le o(\varepsilon\delta\mu),\end{cases}$$

$$\alpha_{15}(t;\varepsilon\delta\mu)=\gamma_1(t;\varepsilon\delta\mu)-\varepsilon\Big[\int_{t_{00}}^{t_0+\tau}Y(\xi;t)f_y[\xi]\dot\varphi_0(\xi-\tau_0)d\xi\Big]dt,$$

(see (4.8),(4.9),(4.11)). Here

$$\sigma_2(\varepsilon\delta\mu)=\int_{t_{00}}^{t_{00}+\tau_0}\Big[\int_0^1\Big|f_x(t,x_0(t)+s\Delta x(t),\varphi_0(t-\tau_0)+s(\varphi_0(t-\tau)-\varphi_0(t-\tau_0))$$

$$+s\delta\varphi(t-\tau),u_0(t)+s\varepsilon\delta u(s))-f_x(t,x_0(t),\varphi_0(t-\tau_0),u_0(t))\Big|ds\Big]dt,\rho_2(\varepsilon\delta\mu)$$

$$=\int_{t_{00}}^{t_{00}+\tau_0}\Big[\int_0^1\Big|f_y(t,x_0(t)+s\Delta x(t),\varphi_0(t-\tau_0)+s(\varphi_0(t-\tau)-\varphi_0(t-\tau_0))$$

$$+s\delta\varphi(t-\tau),u_0(t)+s\varepsilon\delta u(s))-f_y(t,x_0(t),\varphi_0(t-\tau_0),u_0(t))\Big|ds\Big]dt,$$

$$\vartheta_2(\varepsilon\delta\mu)=\int_{t_{00}}^{t_{00}+\tau_0}\Big[\int_0^1\Big|f_u(t,x_0(t)+s\Delta x(t),\varphi_0(t-\tau_0)+s(\varphi_0(t-\tau)-\varphi_0(t-\tau_0))$$

$$+s\delta\varphi(t-\tau),u_0(t)+s\varepsilon\delta u(s))-f_u(t,x_0(t),\varphi_0(t-\tau_0),u_0(t))\Big|ds\Big]dt$$

$$\|Y\|=\sup\Big\{|Y(\xi;t)|:(\xi,t)\in\Pi\Big\},\hat{\gamma}(t;\varepsilon\delta\mu)=\int_{t_{00}}^{t}Y(\xi;t)f_y[\xi]\gamma(\xi;\varepsilon\delta\mu)d\xi.$$

Obviously,

$$\Big|\frac{\hat{\gamma}(t;\varepsilon\delta\mu)}{\varepsilon}\leq\|Y\|\int_{t_{00}}^{t_{00}+\tau_0}|f_y[\xi]|\,\Big|\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|d\xi.$$

By the Lebesguer theorem on passing to the limit under the integral sign, we have

$$\lim_{\varepsilon\to 0}\sigma_2(\varepsilon\delta\mu)=0,\ \lim_{\varepsilon\to 0}\rho_2(\varepsilon\delta\mu)=0,\ \lim_{\varepsilon\to 0}\vartheta_2(\varepsilon\delta\mu)=0,\ \lim_{\varepsilon\to 0}\Big|\frac{\hat{\gamma}(t;\varepsilon\delta\mu)}{\varepsilon}\Big|=0$$

uniformly for $(t,\delta\mu)\in[t_{00},t_{00}+\tau_0]\times V^-$ (see (4.10)). Thus,

$$\alpha_{1i}(t;\varepsilon\delta\mu)=o(t;\varepsilon\delta\mu),i=\overline{1,4}; \tag{4.12}$$

and

$$\alpha_{15}(t;\varepsilon\delta\mu)=-\varepsilon\Big[\int_{t_{00}}^{t_0+\tau}Y(\xi;t)f_y[\xi]\dot{\varphi}_0(\xi-\tau_0)d\xi\Big]\delta\tau+o(t;\varepsilon\delta\mu).$$

Further,

$$\varepsilon\Big[\int_{t_0+\tau}^{t_{00}+\tau_0}Y(\xi;t)f_y[\xi]\dot{\varphi}_0(\xi-\tau_0)d\xi\Big]\delta\tau=o(t;\varepsilon\delta\mu),$$

$$\dot{x}_0(\xi-\tau_0)=\dot{\varphi}_0(\xi-\tau_0),\xi\in[t_{00},t_{00}+\tau_0],$$

therefore,

$$\alpha_{15}(t;\varepsilon\delta\mu)=-\varepsilon\Big[\int_{t_{00}}^{t_{00}+\tau_0}Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau+o(t;\varepsilon\delta\mu). \tag{4.13}$$

On the basis of (4.12) and (4.13) we obtain

$$\alpha_1(t;\varepsilon\delta\mu)=-\varepsilon\Big[\int_{t_{00}}^{t_{00}+\tau_0}Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau+o(t;\varepsilon\delta\mu). \tag{4.14}$$

Now let us transform $\alpha_2(t;\varepsilon\delta\mu)$. We have

$$\alpha_2(t;\varepsilon\delta\mu)=\sum_{i=1}^{4}\alpha_{2i}(t;\varepsilon\delta\mu),$$

where

$$\alpha_{21}(\varepsilon\delta\mu)=\int_{t_0+\tau}^{t_{00}+\tau_0}Y(\xi;t)\Big[f(\xi,x_0(\xi)+\Delta x(\xi),x_0(\xi-\tau)+\Delta x(\xi-\tau),u_0(\xi)+\varepsilon\delta u(\xi))$$

$$-f[\xi]\Big]d\xi,\alpha_{22}(t;\varepsilon\delta\mu)=-\int_{t_0+\tau}^{t_{00}+\tau_0}Y(\xi;t)f_x[\xi]\Delta x(\xi)d\xi,\alpha_{23}(t;\varepsilon\delta\mu)$$

$$=-\int_{t_0+\tau}^{t_{00}+\tau_0}Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi,\alpha_{24}(t;\varepsilon\delta\mu)=-\varepsilon\int_{t_0+\tau}^{t_{00}+\tau_0}Y(\xi;t)f_u[\xi]\Delta\delta u(\xi)d\xi.$$

If $\xi\in[t_0+\tau,t_{00}+\tau_0]$ then

$$|\Delta x(\xi)|\le O(\varepsilon\delta\mu),x_0(\xi-\tau)+\Delta x(\xi-\tau)=x(\xi-\tau;\mu_0+\varepsilon\delta\mu)$$

$$=x_{00}+\varepsilon\delta x_0+\int_{t_0}^{\xi-\tau}f(s,x(s;\mu_0+\varepsilon\delta\mu),x(s-\tau;\mu_0+\varepsilon\delta\mu),u_0(s)+\varepsilon\delta u(s))ds$$

therefore

$$\lim_{\varepsilon\to 0}(\xi,x_0(\xi)+\Delta x(\xi),x_0(\xi-\tau)+\Delta x(\xi-\tau))=(t_{00}+\tau_0,x_0(t_{00}+\tau_0),x_{00})=w_{02}.$$

Moreover,

$$\lim_{\varepsilon\to 0}(\xi,x_0(\xi),x_0(\xi-\tau_0))=(t_{00}+\tau_0,x_0(t_{00}+\tau_0),\varphi_0(t_{00}))=w_{01}.$$

Thus,

$$\lim_{\varepsilon\to 0}\Big[f(\xi,x_0(\xi)+\Delta x(\xi),x_0(\xi-\tau)+\Delta x(\xi-\tau),u_0(\xi)+\varepsilon\delta u(\xi))-f[\xi]\Big]$$

$$=\lim_{(w_1,w_2)\to(w_{01},w_{02})}[f(w_1,u_0(t))-f(w_2,u_0(t))]=f_1^-,w_i\in(a,t_{00}+\tau_0]\times O^2,i=1,2,$$

Since the function $Y(\xi;t)$ is continuous on the set Π, therefore

$$\alpha_{21}(t;\varepsilon\delta\mu)=-\varepsilon Y(t_{00}+\tau_0;t)f_1^-(\delta t_0+\delta\tau)+o(t;\varepsilon\delta\mu).$$

Further, for $\xi\in[t_0+\tau,t_0+\tau_0]$ we have

$$\Delta x(\xi-\tau_0)=\varepsilon\delta\varphi(\xi-\tau_0),$$

therefore

$$\alpha_{23}(t;\varepsilon\delta\mu)=-\varepsilon\int_{t_0+\tau}^{t_0+\tau_0}Y(\xi;t)f_y[\xi]\delta\varphi(\xi-\tau_0)d\xi$$

$$-\int_{t_0+\tau_0}^{t_{00}+\tau_0} Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi = -\int_{t_0+\tau_0}^{t_{00}+\tau_0} Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi$$
$$+o(t;\varepsilon\delta\mu).$$

Obviously,

$$\alpha_{22}(t;\varepsilon\delta\mu)=o(t;\varepsilon\delta\mu), \alpha_{24}(t;\varepsilon\delta\mu)=o(t;\varepsilon\delta\mu).$$

Finally, for $\alpha_2(t;\varepsilon\delta\mu)$ we get

$$\alpha_2(t;\varepsilon\delta\mu) = -\varepsilon Y(t_{00}+\tau_0;t)f_1^-(\delta t_0+\delta\tau) - \int_{t_0+\tau_0}^{t_{00}+\tau_0} Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi$$
$$+o(t;\varepsilon\delta\mu). \tag{4.15}$$

It remains to estimate $\alpha_3(t;\varepsilon\delta\mu)$. We have

$$\alpha_3(t;\varepsilon\delta\mu)=\sum_{i=1}^{5}\alpha_{3i}(t;\varepsilon\delta\mu),\ \alpha_{31}(t;\varepsilon\delta\mu)$$

where

$$\alpha_{31}(t;\varepsilon\delta\mu)=\int_{t_{00}+\tau_0}^{t} Y(\xi;t)\sigma_1(\xi;\varepsilon\delta\mu)\Delta x(\xi)d\xi, \alpha_{32}(t;\varepsilon\delta\mu)$$
$$=\int_{t_{00}+\tau_0}^{t} Y(\xi;t)\rho_1(\xi;\varepsilon\delta\mu)\{x_0(\xi-\tau)-x_0(\xi-\tau_0)+\Delta x(\xi-\tau)\}d\xi,$$
$$\alpha_{33}(t;\varepsilon\delta\mu)=\varepsilon\int_{t_{00}+\tau_0}^{t} Y(\xi;t)\vartheta_1(\xi;\varepsilon\delta\mu)\delta u(\xi)d\xi,\ \alpha_{34}(t;\varepsilon\delta\mu)$$
$$=\int_{t_{00}+\tau_0}^{t} Y(\xi;t)f_y[\xi]\{\Delta x(\xi-\tau)-\Delta x(\xi-\tau_0)\}d\xi,\ \alpha_{35}(t;\varepsilon\delta\mu)$$
$$=\int_{t_{00}+\tau_0}^{t} Y(\xi;t)f_y[\xi]\{x_0(\xi-\tau)-x_0(\xi-\tau_0)\}d\xi.$$

For $\xi\in[t_{00}+\tau_0,t_{10}+\delta_2]$ we have

$$|\Delta x(\xi)|\le O(\varepsilon\delta\mu), |\Delta x(\xi-\tau)|\le O(\varepsilon\delta\mu), \tag{4.16}$$

(see (4.2)). For each fixed Lebesgue point $\xi\in(t_{00}+\tau_0,t_{10}+\delta_2)$ of function $\dot x_0(\xi-\tau_0)$ we get

$$x_0(\xi-\tau)-x_0(\xi-\tau_0)=\int_{\xi}^{\xi-\varepsilon\delta\tau}\dot x_0(s-\tau_0)ds$$
$$=-\varepsilon\dot x_0(\xi-\tau_0)\delta\tau+\gamma_1(\xi;\varepsilon\delta\mu), \tag{4.17}$$

with

$$\lim_{\varepsilon\to 0}\frac{\gamma_1(\xi;\varepsilon\delta\mu)}{\varepsilon}=0 \text{ uniformly for } \delta\mu\in V^-. \tag{4.18}$$

Thus, (4.17) and (4.18) are valid for almost all points of the interval $(t_{00}+\tau_0, t_{10}+\delta_2)$. From (4.17) taking into account boundedness of the function $f(t,x,y,u)$ it follows

$$| x_0(\xi-\tau) - x_0(\xi-\tau_0) | \leq O(\varepsilon\delta\mu) \text{ and } \Big|\frac{\gamma_1(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big| \leq const. \tag{4.19}$$

For $\alpha_{3i}(t;\varepsilon\delta\mu)), i=\overline{1,4}$ we have

$$\begin{cases} | \ \alpha_{31}(t;\varepsilon\delta\mu)| \leq \| \ Y \ \| \ O(\varepsilon\delta\mu)\sigma_3(\varepsilon\delta\mu), \\ | \ \alpha_{32}(t;\varepsilon\delta\mu)| \leq \| \ Y \ \| \ O(\varepsilon\delta\mu)\rho_3(\varepsilon\delta\mu), \\ | \ \alpha_{33}(t;\varepsilon\delta\mu)| \leq \varepsilon \ \| \ Y \ \| \ \vartheta_3(\varepsilon\delta\mu), \\ | \ \alpha_{34}(t;\varepsilon\delta\mu)| \leq o(\varepsilon\delta\mu), \end{cases}$$

(see (4.17),(4.19) and Lemma 2.4). Here

$$\sigma_3(\varepsilon\delta\mu) = \int_{t_{00}+\tau_0}^{t_{10}+\delta_2} \sigma_1(\xi;\varepsilon\delta\mu)d\xi, \rho_3(\varepsilon\delta\mu) = \int_{t_{00}+\tau_0}^{t_{10}+\delta_2} \rho_1(\xi;\varepsilon\delta\mu)d\xi,$$

$$\vartheta_3(\varepsilon\delta\mu) = \int_{t_{00}+\tau_0}^{t_{10}+\delta_2} \vartheta_1(\xi;\varepsilon\delta\mu)d\xi.$$

Obviously,

$$\Big|\frac{\hat{\gamma}_1(t;\varepsilon\delta\mu)}{\varepsilon} \leq \| \ Y \ \| \int_{t_{00}+\tau_0}^{t_{10}+\delta_2} |f_y[\xi]|\Big|\frac{\gamma_1(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|d\xi.$$

By the Lebesguer theorem on passing to the limit under the integral sign, we have

$$\lim_{\varepsilon\to 0}\sigma_3(\varepsilon\delta\mu)=0, \ \lim_{\varepsilon\to 0}\rho_3(\varepsilon\delta\mu)=0, \ \lim_{\varepsilon\to 0}\vartheta_3(\varepsilon\delta\mu)=0, \ \lim_{\varepsilon\to 0}\Big|\frac{\hat{\gamma}_1(t;\varepsilon\delta\mu)}{\varepsilon}\Big|=0$$

uniformly for $(t,\delta\mu)\in[t_{00}, t_{10}+\delta_2]\times V^-$ (see (4.18)).
Thus,

$$\alpha_{3i}(t;\varepsilon\delta\mu) = o(t;\varepsilon\delta\mu), i=\overline{1,4}$$

and

$$\alpha_{35}(t;\varepsilon\delta\mu) = -\varepsilon\Big[\int_{t_{00}+\tau_0}^{t} Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau + o(t;\varepsilon\delta\mu).$$

On the basis of last relations we get

$$\alpha_3(t;\varepsilon\delta\mu) = -\varepsilon\Big[\int_{t_{00}}^{t_{00}+\tau_0} Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau + o(t;\varepsilon\delta\mu). \tag{4.20}$$

Taking into account (4.14),(4.15) and (4.20) the expression (4.7) can be represented in the form

$$R_1(t;t_{00},\varepsilon\delta\mu) = -\varepsilon Y(t_{00}+\tau_0;t)f_1^-\delta t_0 - \varepsilon\Big[f_1^- + \int_{t_{00}}^{t} Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau$$

$$-\int_{t_0+\tau_0}^{t_{00}+\tau_0} Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi + o(t;\varepsilon\delta\mu). \qquad (4.21)$$

Finally, from (4.3) by virtue of (4.6) and (4.21) we obtain (3.1), where $\delta x(t;\delta\mu)$ has the form (3.2).

5. Proof of Theorem 3.2

The function $\Delta x(t)$ satisfies equation (4.1) on the interval $[t_0, t_{10}+\delta_2]$. By using the Cauchy formula, we can represent it in the form

$$\Delta x(t) = Y(t_0;t)\Delta x(t_0) + \varepsilon\int_{t_0}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)d\xi + \sum_{i=0}^{1} R_i(t;t_0,\varepsilon\delta\mu), \qquad (5.1)$$

(see (4.4)). Let a number $\delta_2 \in (0,\delta_1]$ be so small that $t_{00}+\tau_0 < t_{10}-\delta_2$. The matrix function $Y(\xi;t)$ is continuous on Π, therefore

$$Y(t_{00};t)\Delta x(t_{00}) = \varepsilon Y(t_{00};t)\Big[\delta x_0 - f^+\delta t_0\Big] + o(t;\varepsilon\delta\mu) \qquad (5.2)$$

(see (2.8)).
Now let us transform $R_0(t;t_0,\varepsilon\delta\mu)$. It is not difficult to see that

$$R_0(t;t_0,\varepsilon\delta\mu) = \varepsilon\int_{t_0-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$

$$+\int_{t_{00}}^{t_0} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\Delta x(\xi)d\xi = \varepsilon\int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$

$$+\int_{t_{00}+\tau_0}^{t_0+\tau_0} Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi + o(t;\varepsilon\delta\mu). \qquad (5.3)$$

In a similar way, for $t \in [t_{10}-\delta_2, t_{10}+\delta_2]$ one can prove

$$R_1(t;t_0,\varepsilon\delta\mu) = -\varepsilon Y(t_{00}+\tau_0;t)f_1^+\delta t_0 - \varepsilon\Big[f_1^+ + \int_{t_{00}}^{t} Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau$$

$$-\int_{t_{00}+\tau_0}^{t_0+\tau_0} Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi + o(t;\varepsilon\delta\mu). \qquad (5.4)$$

Finally, we note that

$$\varepsilon\int_{t_0}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)d\xi = \varepsilon\int_{t_{00}}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)d\xi + o(t;\varepsilon\delta\mu). \qquad (5.5)$$

for $t \in [t_{10}-\delta_2, t_{10}+\delta_2]$.

Taking into account (5.2)-(5.5), from (5.1), we obtain (3.1), where $\delta x(t;\varepsilon\delta\mu)$ has form (3.6).

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Authors' addresses:

T. Tadumadze
I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: tamaztad@yahoo.com,
tamaz.tadumadze@tsu.ge

N. Gorgodze
A. Tsereteli Kutaisi State University
59, Tamap Mepe St., Kutaisi 4600
Georgia
E-mail: nika_gorgodze@yahoo.com