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EXISTENCE OF OPTIMAL INITIAL DATA AND CONTINUITY OF THE INTEGRAL FUNCTIONAL MINIMUM WITH RESPECT TO PERTURBATIONS FOR A CLASS OF NEUTRAL DIFFERENTIAL EQUATION

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Abstract. For the system of differential equations, linear with respect to prehistory of velocity, sufficient conditions of existence of optimal initial data are obtained. Under initial data we imply the collection of constant delays, initial moment and vector, initial functions. The question of the continuity of the integral functional minimum with respect to perturbations of the right-hand side of equation and integrand is investigated.

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Let $0 < \tau_{1i} < \tau_{2i}, i = \overline{1,s}, 0 < \eta_{1j} < \eta_{2j}, j = \overline{1,m}, t_1 < t_2 < t_3$ be given numbers, with $t_3 - t_2 > \tau = \max(\tau_{21}, ..., \tau_{2s}, \eta_{21}, ..., \eta_{2m})$; *let* R^n be the n-dimensional vector space of points

$$x = (x^1, ..., x^n)^T, |x|^2 = \sum_{i=1}^{n} (x^i)^2,$$

where T means transpose; the functions

$$F_i(t,x,y) = (f_i^0(t,x,y), f_i(t,x,y))^T \in R^{1+n}, i = \overline{1,s}$$

are continuous on the set $I \times R^n \times R^n$, where $I = [t_1, t_3]$, and continuously differentiable with respect to $x, y \in R^n$; suppose that $\Phi \subset R^n, X_0 \subset R^n$ are compact sets, $V \subset R^n$ is a compact and convex set. By Δ_1 and Δ_2 we denote sets of measurable $\varphi(t) \in \Phi, t \in I_1 = [\hat{\tau}, t_2], \hat{\tau} = a - \tau$, and $v(t) \in V, t \in I_1$ initial functions, respectively. Further, $R^{n\times n}$ is the space of matrices

$$A = (a_{ij})_{i,j=1}^{n}, |A|^2 = \sum_{i,j=1}^{n} (a_{ij})^2;$$

the functions $A_j(t) \in R^{n\times n}, a_j(t) = (a_j^1(t), ..., a_j^n(t)), j = \overline{1,m}$ are continuous on the interval I.

The collection of initial data $\tau_i, i = \overline{1,s}, \eta_j, j = \overline{1,m}, t_0, x_0, \varphi(t), v(t)$ is said to be initial element and we denote it by w.

To each initial element

$$w = (\tau_1, ..., \tau_s, \eta_1, ..., \eta_m, t_0, x_0, \varphi, v) \in W = [\tau_{11}, \tau_{21}] \times \cdots \times [\tau_{1s}, \tau_{2s}]$$

$$\times [\eta_{11}, \eta_{21}] \times ... \times [\eta_{1m}, \eta_{2m}] \times [t_1, t_2] \times X_0 \times \Delta_1 \times \Delta_2$$

we assign the neutral differential equation

$$\dot{x}(t)=\sum_{i=1}^{s} f_i(t,x(t),x(t-\tau_i))+\sum_{j=1}^{m} A_j(t)\dot{x}(t-\eta_j), t\in[t_0,t_3] \tag{1}$$

with the initial condition

$$x(t)=\varphi(t), \dot{x}(t)=v(t), t\in[\hat{\tau},t_0), x(t_0)=x_0. \tag{2}$$

Remark. The symbol $\dot{x}(t)$ on the interval $[\hat{\tau},t_0)$ is not connected with derivative of the function $\varphi(t)$.

Definition 1. Let $w\in W$. A function $x(t)=x(t;w)\in R^n, t\in[\hat{\tau},t_3]$ is called a solution, corresponding to the element w, if it satisfies condition (2), is absolutely continuous on the interval $[t_0,t_3]$ and satisfies Eq.(1) almost everywhere on $[t_0,t_3]$.

By W_0 we denote the set of such initial elements $w\in W$ for which there exists the corresponding solution $x(t;w)$. In what follows we will assume that $W_0\neq\emptyset$.

We note that, if the following condition

$$|f_x(t,x,y)|+|f_y(t,x,y)|\leq L, \forall(t,x,y)\in I\times R^n\times R^n$$

is fulfilled, where $L>0$ is a given number, then $W_0=W$.

Let us consider the following functional

$$J(w)=\sum_{i=1}^{s}\int_{t_0}^{t_3}[f_i^0(t,x(t),x(t-\tau_i))$$
$$+\sum_{j=1}^{m} a_j(t)\dot{x}(t-\eta_j)]dt, w\in W_0,$$

where $x(t)=x(t;w)$.

Definition 2. An initial element $w_0=(\tau_{10},...,\tau_{s0},\eta_{10},...,\eta_{m0},t_{00},x_{00},\varphi_0,v_0)\in W_0$ is said to be optimal for the differential equation (1) if

$$J(w_0)\leq J(w)$$

for any $w\in W_0$.

Theorem 1. *Let the following conditions hold:*

1) *there exists a compact $K_0\subset R^n$ such that*

$$x(t;w)\in K_0, t\in[t_0,t_3], \forall w\in W_0;$$

2) *for any $(\xi_i,x_i)\in I\times K_0, i=\overline{1,s}$ the set*

$$\big\{(F_1(\xi_1,x_1,y),...,F_s(\xi_s,x_s,y)): y\in\Phi\big\}\subset R^{(1+n)s}$$

is convex. Then there exists an optimal initial element w_0.

Theorem 2. *Let $f_i(t,x,y)=B_i(t,x)y, B(t,x)\in R^{n\times n}$ and function $f_i^0(t,x,y)$ is convex with respect to y. Moreover, let the set Φ be convex and let the condition 1) of Theorem 1 be fulfilled. Then there exists an optimal initial element w_0.*

Theorems 1,2 are proved by a scheme given in [1,2].

Let us consider the perturbed differential equation

$$\dot{x}(t)=\sum_{i=1}^{s}\left[f_i(t,x(t),x(t-\tau_i))+g_{i\delta}(t,x(t),x(t-\tau_i))\right]$$

$$+\sum_{j=1}^{m}[A_j(t)+A_{j\delta}(t)]\dot{x}(t-\eta_j),t\in[t_0,t_3] \qquad (3)$$

with the initial condition (2) and the perturbed functional

$$J(w;\delta)=\int_{t_0}^{t_3}\Big\{\sum_{i=1}^{s}\left[f_i^0(t,x(t),x(t-\tau_i))+g_{i\delta}^0(t,x(t),x(t-\tau_i))\right.$$

$$\left.+\sum_{j=1}^{m}\left[a_j(t)+a_{j\delta}(t)\right]\dot{x}(t-\eta_j)\right\}dt,$$

where the functions $G_{i\delta}(t,x,y)=(g_{i\delta}^0(t,x,y),g_{i\delta}(t,x,y))^T, i=\overline{1,s}$ are continuous on the set $I\times R^n\times R^n$ and continuously differentiable with respect to $x,y\in R^n$; $A_{j\delta}(t),a_{j\delta}(t)$, $j=\overline{1,m},t\in I$ are continuous functions.

Definition 3. An initial element $w_{0\delta}=(\tau_{1\delta},...,\tau_{s\delta},\eta_{1\delta},...,\eta_{m\delta},t_{0\delta},x_{0\delta},\varphi_\delta,v_\delta)\in W_0$ is said to be optimal for the differential equation (3) if

$$J(w_{0\delta};\delta)\le J(w;\delta)$$

for any $w\in W_0$.

Theorem 3. *Let the conditions of the Theorem 1 hold . Then for any $\varepsilon>0$ there exists a number $\delta>0$ such that for arbitrary functions $G_{i\delta}(t,x,y), i=\overline{1,s}$; $A_{j\delta}(t),a_{j\delta}(t)$, $j=\overline{1,m}$ satisfying the conditions:*

$$\sum_{i=1}^{s}\int_{t_1}^{t_3}\sup\Big\{\Big|\frac{\partial G_{i\delta}(t,x,y)}{\partial x}\Big|+\Big|\frac{\partial G_{i\delta}(t,x,y)}{\partial y}\Big|:x,y\in K_1\Big\}dt\le C, \qquad (4)$$

$$\sum_{i=1}^{s}\int_{t_1}^{t_3}\sup\Big\{\Big|G_{i\delta}(t,x,y)\Big|:x,y\in K_1\Big\}dt+\sum_{j=1}^{m}\Big[\|A_{j\delta}\|+\|a_{j\delta}\|\Big]\le\delta \qquad (5)$$

and the set

$$\Big\{(F_1(\xi_1,x_1,y)+G_{1\delta}(\xi_1,x_1,y),...,F_s(\xi_s,x_s,y)+G_{s\delta}(\xi_s,x_s,y)):y\in\Phi\Big\}$$

is convex, where $C>0$ is a fixed number,

$$\|A_{j\delta}\|=\sup\{|A_{j\delta}(t)|:t\in I\}$$

and $K_1\subset R^n$ is a compact set containing some neighborhood of set $K_0\cup\Phi$; there exists an optimal initial element $w_{0\delta}$ and the following inequality

$$|J(w_{0\delta};\delta)-J(w_0)|\le\varepsilon \qquad (6)$$

is fulfilled.

Theorem 4. *Let the conditions of Theorem 2 hold. Then for any* $\varepsilon > 0$ *there exists a number* $\delta > 0$ *such that for arbitrary functions*

$$G_{i\delta}(t,x,y) = (g^0_{i\delta}(t,x,y), B_{i\delta}(t,x)y)^T, i = \overline{1,s}$$

and $A_{j\delta}(t), a^0_{j\delta}(t), j = \overline{1,m}$ *satisfying the conditions (4),(5) and the functions* $g^0_{i\delta}(t,x,y)$, $i = \overline{1,s}$ *are convex with respect to* y; *there exists an optimal element* $w_{0\delta}$ *and the inequality (6) is fulfilled.*

Theorems 3,4 are proved by a scheme given in [3]. The similar questions are considered for delay differential equations in [4].

Finally, we note that Theorems 1-4 play an important role in solving inverse problems for neutral differential equations [5].

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