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NECESSARY OPTIMALITY CONDITIONS OF SECOND ORDER IN DISCRETE TWO-PARAMETER STEPWISE CONTROL PROBLEMS

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Abstract. A stepwise optimal control problem described by two-dimensional discrete systems is considered. Under openness of a control domain, necessary optimality conditions of first and second order are obtained.

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1. Introduction

Discrete dynamical models of controlled systems are an important class among of mathematical models. Such models arise in modeling of real processes and discretization of continuous models [1-10]. Optimization problems of stepwise or variable structure systems occupy an important place in the theory of optimal control [11-21]. The present paper is devoted to derivation of necessary optimality conditions for one class of control problem described by two-dimensional stepwise discrete system. Finally, we note that various necessary and sufficient optimality conditions for discrete two-dimensional controlled systems are obtained in [8, 22-27].

2. Statement of the problem

Let the controlled system be described by the following discrete two-parametric system of equations

$$\begin{cases} z_i(t+1,x+1) = f_i(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x)), \\ (t,x) \in D_i, i=\overline{1,3}, \end{cases} \tag{2.1}$$

with boundary conditions

$$\begin{cases} z_1(t_0,x) = a_1(x),\ x=x_0,x_0+1,...,X,\ z_1(t,x_0)=\beta_1(t),\ t=t_0,t_0+1,...,t_1, \\ z_2(t_1,x) = z_1(t_1,x),\ x=x_0,x_0+1,...,X,\ z_2(t,x_0)=\beta_2(t),\ t=t_1,t_1+1,...,t_2, \\ z_3(t_2,x) = z_2(t_2,x),\ x=x_0,x_0+1,...,X,\ z_3(t,x_0)=\beta_3(t),\ t=t_2,t_2+1,...,t_3, \\ a_1(x_0)=\beta_1(t_0),\ z_1(t_1,x_0)=\beta_2(t_1),\ z_2(t_2,x_0)=\beta_3(t_2). \end{cases} \tag{2.2}$$

Here, $D_i = \{(t,x) : t = t_{i-1}, t_{i-1}+1, ..., t_i - 1; x = x_0, x_0+1, ..., X-1\}, i=\overline{1,3}$, where $x_0, X, t_i, i=\overline{1,3}$ are fixed numbers; $f_i(t,x,a_i,b_i,u_i), i=\overline{1,3}$ are n-dimensional vector-functions continuous in the aggregate of variables together with partial derivatives with respect to $(z_i,a_i,b_i,u_i), i=\overline{1,3}$ up to the second order inclusive, $\alpha_1(x), \beta_i(t), i=\overline{1,3}$

are given n-dimensional vector-functions, and $u_i(t,x), i=\overline{1,3}$ are r-dimensional control functions with values from the given non-empty, bounded and open sets $U_i \subset R^r, i=\overline{1,3}$, i.e.

$$u_i(t,x) \in U_i \subset R^r, \quad (t,x) \in D_i, \quad i=\overline{1,3}. \tag{2.3}$$

The triple $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))'$ with the above mentioned properties and its corresponding solution $z(t,x) = (z_1(t,x), z_2(t,x), z_3(t,x))'$ of boundary value problem (2.1)-(2.2) will be called an admissible control and admissible state of the process, respectively. The pair $(u(t,x), z(t,x))$ is said to be an admissible process.

The problem is to minimize the cost functional

$$S(u) = \sum_{i=1}^{3} \varphi_i(z_i(t_i, X)) \tag{2.4}$$

determined on the solutions of boundary value problem (2.1)-(2.2) generated by all possible admissible controls.
Here, $\varphi_i(z_i), i=\overline{1,3}$ are the given twice continuously differentiable scalar functions.
In the sequel, the problem on the minimum of the functional (2.4) under restrictions (2.1)-(2.3) will be called problem (2.1)-(2.4). The admissible process $(u(t,x), z(t,x))$ being a solution of problem (2.1)-(2.4) will be called an optimal process.

3. Auxiliary facts and variations of cost functional

Let $(u(t,x), z(t,x))$ be a fixed admissible process. In the sequel, the following denotations will be used:

$$H_i(t,x,z_i,a_i,b_i,u_i,\psi_i) = \psi_i' f_i(t,x,z_i,a_i,b_i,u_i),$$

$$\frac{\partial f_i[t,x]}{\partial a_i} = \frac{\partial f_i(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x))}{\partial a_i},$$

$$\frac{\partial H_i[t,x]}{\partial z_i} = \frac{\partial H_i(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x),\psi_i(t,x))}{\partial z_i},$$

$$\frac{\partial^2 H_i[t,x]}{\partial z_i^2} = \frac{\partial^2 H_i(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x),\psi_i(t,x))}{\partial z_i^2},$$

where $\psi_i = \psi_i(t,x), \ i=\overline{1,3}$ are n-dimensional vector-functions of conjugated being the solutions of the problem

$$\psi_i(t-1,x-1) = \frac{\partial H_i[t,x]}{\partial z_i} + \frac{\partial H_i[t-1,x]}{\partial a_i} + \frac{\partial H_i[t,x-1]}{\partial b_i}, \quad i=\overline{1,3}, \tag{3.1}$$

$$\psi_1(t_1-1,X-1) = \psi_2(t_1-1,X-1) - \frac{\partial \varphi_1(z_1(t_1,X))}{\partial z_1},$$

$$\psi_1(t_1-1,x-1) = \psi_2(t_1-1,x-1) + \frac{\partial H_1[t_1-1,x]}{\partial a_1} - \frac{\partial H_2[t_1-1,x]}{\partial a_2},$$

$$\psi_1(t-1,X-1) = \frac{\partial H_1[t-1,X-1]}{\partial b_1},$$

$$\psi_2(t_2-1,X-1) = \psi_3(t_2-1,X-1) - \frac{\partial \varphi_2(z_2(t_2,X))}{\partial z_2},$$

$$\psi_2(t_2-1,x-1)=\psi_3(t_2-1,x-1)+\frac{\partial H_2[t_2-1,x]}{\partial a_2}-\frac{\partial H_3[t_2-1,x]}{\partial a_3},$$
$$\psi_2(t-1,X-1)=\frac{\partial H_2[t,X-1]}{\partial b_2},\quad \psi_3(t_3-1,X-1)=-\frac{\partial \varphi_3(z_3(t_3,X))}{\partial z_3},$$
$$\psi_3(t_3-1,x-1)=\frac{\partial H_3[t_3,x]}{\partial a_3},\quad \psi_3(t-1,X-1)=\frac{\partial H_3[t,X-1]}{\partial b_3}. \tag{3.2}$$

Using a scheme for example from [23, 28, 29] we can show that the first and second variations (in the classical sense) of functional (2.4) have the form

$$\delta^1 S(u;\,\delta u)=-\sum_{i=1}^{3}\left[\sum_{t=t_{i-1}}^{t_1-1}\sum_{x=x_0}^{X-1}\frac{\partial H_i^{'}[t,x]}{\partial u_i}\delta u_i(t,x)\right], \tag{3.3}$$

$$\delta^2 S(u;\,\delta u)=\sum_{i=1}^{3}\delta z_i^{'}(t_i,X)\frac{\partial^2\varphi_i(z_i(t_i,X)}{\partial z_i^2}\delta z_i(t_i,X))$$
$$-\sum_{i=1}^{3}\left[\sum_{t=t_{i-1}}^{t_1-1}\sum_{x=x_0}^{X-1}\left[\delta z_i^{'}(t,x)\frac{\partial^2 H_i[t,x]}{\partial z_i^2}\delta z_i(t,x)+\delta z_i^{'}(t+1,x)\frac{\partial^2 H_i[t,x]}{\partial a_i\,\partial z_i}\delta z_i(t,x)+\delta z_i^{'}(t,x)\right.\right.$$
$$\times\frac{\partial^2 H_i[t,x]}{\partial z_i\,\partial a_i}\delta z_i(t+1,x)+\delta z_i^{'}(t+1,x)\frac{\partial^2 H_i[t,x]}{\partial a_i^2}\delta z_i(t+1,x)+\delta z_i^{'}(t,x)\frac{\partial^2 H_i[t,x]}{\partial z_i\,\partial b_i}$$
$$\times\delta z_i(t,x+1)+\delta z_i^{'}(t,x+1)\frac{\partial^2 H_i[t,x]}{\partial b_i\,\partial z_i}\delta z_i(t,x)+\delta z_i^{'}(t,x+1)\frac{\partial^2 H_i[t,x]}{\partial b_i^2}\delta z_i(t,x+1)$$
$$+\delta z_i^{'}(t+1,x)\frac{\partial^2 H_i[t,x]}{\partial a_i\,\partial b_i}\delta z_i(t,x+1)+\delta z_i^{'}(t,x+1)\frac{\partial^2 H_i[t,x]}{\partial b_i\,\partial a_i}\delta z_i(t+1,x)$$
$$+2\delta u_i^{'}(t,x)\frac{\partial^2 H_i[t,x]}{\partial u_i\,\partial z_i}\delta z_i(t,x)+2\delta u_i^{'}(t,x)\frac{\partial^2 H_i[t,x]}{\partial u_i\,\partial a_i}\delta z_i(t+1,x)+2\delta u_i^{'}(t,x)\frac{\partial^2 H_i[t,x]}{\partial u_i\,\partial b_i}$$
$$\left.\left.\times\delta z_i(t,x+1)+\delta u_i^{'}(t,x)\frac{\partial^2 H_i[t,x]}{\partial u_i^2}\delta u_i(t,x)\right]\right] \tag{3.4}$$

respectively, where $\delta u_i(t,x)\in R^r$, $(t,x)\in D_i$, $i=\overline{1,3}$ is an arbitrary bounded vector-function called an admissible variation of the control $u_i(t,x)$, $i=\overline{1,3}$, and $\delta z_i(t,x)$ is a variation of the trajectory $z_i(t,x)$ being a solution of the equation in variations

$$\delta z_i(t+1,x+1)=\frac{\partial f_i[t,x]}{\partial z_i}\delta z_i(t,x)+\frac{\partial f_i[t,x]}{\partial a_i}\delta z_i(t+1,x)+\frac{\partial f_i[t,x]}{\partial b_i}\delta z_i(t,x+1)$$
$$+\frac{\partial f_i[t,x]}{\partial u_i}\delta u_i(t,x),\quad i=\overline{1,3}, \tag{3.5}$$

with boundary conditions

$$\begin{cases}\delta z_1(t_0,x)=0,\ x=x_0,x_0+1,...,X,\ \delta z_1(t,x_0)=0,\ t=t_0,t_0+1,...,t_1,\\ \delta z_2(t_1,x)=\delta z_1(t_1,x),\ x=x_0,x_0+1,...,X,\ \delta z_2(t,x)=0,\ t=t_1,t_1+1,...,t_2,\\ \delta z_3(t_2,x)=\delta z_2(t_2,x),\ x=x_0,x_0+1,...,X,\ \delta z_3(t,x_0)=0,\ t=t_2,t_2+1,...,t_3.\end{cases} \tag{3.6}$$

The system of difference equations (3.5) is linear and nonhomogeneous. Therefore, we can represent (see [23, 26, 27]) the solution of problem (3.5)-(3.6) in the form

$$\delta z_1(t,x)=\sum_{\tau=t_0}^{t-1}\sum_{s=x_0}^{x-1}R_1(t,x;\tau,s)\frac{\partial f_1[\tau,s]}{\partial u_1}\delta u_1(\tau,s), \tag{3.7}$$

$$\delta z_2(t,x)=\sum_{\tau=t_0}^{t_1-1}\sum_{s=x_0}^{x-1}Q_1(t,x;\tau,s)\frac{\partial f_1[\tau,s]}{\partial u_1}\delta u_1(\tau,s)$$
$$+\sum_{\tau=t_1}^{t-1}\sum_{s=x_0}^{x-1}R_2(t,x;\tau,s)\frac{\partial f_2[\tau,s]}{\partial u_2}\delta u_2(\tau,s), \tag{3.8}$$

$$\delta z_3(t,x)=\sum_{\tau=t_0}^{t_1-1}\sum_{s=x_0}^{x-1}Q_2(t,x;\tau,s)\frac{\partial f_1[\tau,s]}{\partial u_1}\delta u_1(\tau,s)+\sum_{\tau=t_1}^{t_2-1}\sum_{s=x_0}^{x-1}Q_3(t,x;\tau,s)$$
$$\times\frac{\partial f_2[\tau,s]}{\partial u_2}\delta u_2(\tau,s)+\sum_{\tau=t_2}^{t-1}\sum_{s=x_0}^{x-1}R_3(t,x;\tau,s)\frac{\partial f_3[\tau,s]}{\partial u_3}\delta u_3(\tau,s), \tag{3.9}$$

where by definition

$$Q_1(t,x;\tau,s)=R_2(t,x;t_1-1,x-1)\,R_1(t,x;\tau,s)$$
$$+\sum_{\beta=s+1}^{x-1}\left[R_2(t,x;t_1-1,\beta-1)-R_2(t,x;t_1-1,\beta)\frac{\partial f_2[t_1-1,\beta]}{\partial a_2}\right]\,R_1(t_1,\beta;\,\tau,s),$$

$$Q_2(t,x;\tau,s)=R_3(t,x;t_2-1,x-1)\,Q_1(t_2,x;\tau,s)$$
$$+\sum_{\beta=s+1}^{x-1}\left[[R_3(t,x;t_2-1,\beta-1)-R_3(t,x;t_2-1,\beta)]\,\frac{\partial f_3[t_2-1,\beta]}{\partial a_3}\right]\,Q_1(t_2,\beta;\,\tau,s),$$

$$Q_3(t,x;\tau,s)=R_3(t,x;t_2-1,x-1)\,R_2(t_2,x;\tau,s)$$
$$+\sum_{\beta=s+1}^{x-1}\left[[R_3(t,x;t_2-1,\beta-1)-R_3(t,x;t_2-1,\beta)]\,\frac{\partial f_3[t_2-1,\beta]}{\partial a_3}\right]\,R_2(t_2,\beta;\,\tau,s),$$

Here, $R_i(t,x;\tau,s)$, $i=\overline{1,3}$ are $(n\times n)$ dimensional matrix functions being the solutions of the following problems:

$$R_i(t,x;\tau-1,s-1)=R_i(t,x;\tau,s)\frac{\partial f_i[\tau,s]}{\partial z_i}+R_i(t,x;\tau-1,s)\frac{\partial f_i[\tau-1,s]}{\partial a_i}$$
$$+R_i(t,x;\tau,s-1)\frac{\partial f_i[\tau,s-1]}{\partial b_i},$$
$$R_i(t,x;t-1,s-1)=R_i(t,x;t_1-1,s)\frac{\partial f_i[t-1,s]}{\partial a_i},$$
$$R_i(t,x;\tau-1,x-1)=R_i(t,x;\tau,x-1)\frac{\partial f_i[\tau,x-1]}{\partial b_i},$$
$$R_i(t,x;t-1,x-1)=E,\ (E-(n\times n)\text{ is a unit function}).$$

Let $(u(t,x), z(t,x))$ be an optimal process. Then, along this process, for all the admissible variations $\delta u(t,x)$ of the control $u(t,x)$, the first variation (3.3) of functional (2.4) should equal zero, the second variation (3.4) of functional (2.4) should be non-negative, i.e.

$$\delta^1 S(u;\, \delta u) = 0, \tag{3.10}$$

$$\delta^2 S(u;\, \delta u) \geq 0. \tag{3.11}$$

The relations (3.10) and (3.11) are implicit necessary conditions of first and second orders, respectively.

In the next section, using these relations we obtain the explicit necessary optimality conditions expressed directly by the parameter of the stated problem.

4. Necessary optimality conditions

Allowing for representation (3.10), by independence of the admissible variations $\delta u_i(t,x)$, $i = \overline{1,3}$ of the control it follows from (3.3) that along the optimal process

$$\frac{\partial H_i[\theta,\xi]}{\partial u_i} = 0, \quad \text{for all } \ (\theta,\xi) \in D_i, \ i = \overline{1,3}. \tag{4.1}$$

The relation (4.1) representing a first order necessary optimality conditions is an analogy of Euler equation for problem (2.1)-(2.4).

Each admissible control $u(t,x)$ satisfying Euler equation (4.1) is said to be classic extremal in problem (2.1)-(2.4).

Using inequality (3.11), in many cases we can get explicit necessary optimality condition of second order.

To this end, assume that in system (2.1)

$$f_i(t,x,z_i,a_i,b_i,u_i) = B_i(t,x)\, b_i + F(t,x,z_i,a_i,u_i). \tag{4.2}$$

Assume

$$K_1(\tau,s) = -R_1'(t_1,X;\,\theta,\tau)\,\frac{\partial^2\varphi_1(z_1(t_1,X))}{\partial z_1^2}\,R_1(t_1,X;\,\theta,s) - Q_1'(t_2,X;\,\theta,\tau)$$

$$\times\frac{\partial^2\varphi_2(z_2(t_2,X))}{\partial z_2^2}\,Q_2(t_2,X;\,\theta,s) - Q_3'(t_3,X;\,\theta,\tau)\,\frac{\partial^2\varphi_3(z_3(t_3,X))}{\partial z_3^2}\,Q_3(t_3,X;\,\theta,s)$$

$$+\sum_{t=\theta+1}^{t_1-1}\ \sum_{x=max(\tau,s)+1}^{X-1}\Bigg[R_1'(t,x;\,\theta,\tau)\,\frac{\partial^2 H_1[t,x]}{\partial z_1^2}\,R_1(t,x;\,\theta,s) + R_1'(t,x;\,\theta,\tau)\,\frac{\partial^2 H_1[t,x]}{\partial z_1\,\partial a_1}$$

$$\times R_1(t+1,x;\,\theta,s) + R_1'(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_1[t,x]}{\partial a_1\,\partial z_1}\,R_1(t,x;\,\theta,s)\Bigg]$$

$$+\sum_{t=\theta}^{t_1-1}\ \sum_{x=max(\tau,s)+1}^{X-1} R_1'(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_1[t,x]}{\partial a_1^2}\,R_1(t+1,x;\,\theta,s)$$

$$+\sum_{t=t_1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}\Bigg[Q_1^{'}(t,x;\,\theta,\tau)\,\frac{\partial^2 H_2[t,x]}{\partial z_2^2}\,Q_1(t,x;\,\theta,s)+Q_1^{'}(t,x;\,\theta,\tau)\,\frac{\partial^2 H_2[t,x]}{\partial z_2\,\partial a_2}$$

$$\times Q_1(t+1,x;\,\theta,s)+Q_1^{'}(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_2[t,x]}{\partial a_2\partial z_2}\,Q_1(t,x;\,\theta,s)+Q_1^{'}(t+1,x;\,\theta,\tau)$$

$$\times\frac{\partial^2 H_2[t,x]}{\partial a_2^2}\,Q_1(t+1,x;\,\theta,s)\Bigg]+\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}\Bigg[Q_2^{'}(t,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial z_3^2}\,Q_2(t,x;\,\theta,s)$$

$$+Q_2^{'}(t,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial z_3\,\partial a_3}\,Q_2(t+1,x;\,\theta,s)+Q_2^{'}(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial a_3\,\partial z_3}\,Q_2(t,x;\,\theta,s)$$

$$+Q_2^{'}(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial a_3^2}\,Q_2(t+1,x;\,\theta,s)\Bigg], \tag{4.3}$$

$$K_2(\tau,s)=-R_2^{'}(t_2,X;\,\theta,\tau)\,\frac{\partial^2\varphi_2(z_2(t_2,X))}{\partial z_2^2}\,R_2(t_2,X;\,\theta,s)-Q_3^{'}(t_3,X;\,\theta,\tau)$$

$$\times\frac{\partial^2\varphi_3(z_3(t_3,X))}{\partial z_3^2}\,Q_3(t_3,X;\,\theta,s)+\sum_{t=\theta+1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}\Bigg[R_2^{'}(t,x;\,\theta,\tau)\,\frac{\partial^2 H_2[t,x]}{\partial z_2^2}$$

$$\times R_2(t,x;\,\theta,s)+R_2^{'}(t,x;\,\theta,\tau)\,\frac{\partial^2 H_2[t,x]}{\partial z_2\,\partial a_2}\,R_2(t+1,x;\,\theta,s)+R_2^{'}(t+1,x;\,\theta,\tau)$$

$$\times\frac{\partial^2 H_2[t,x]}{\partial a_2\,\partial z_2}\,R_2(t,x;\,\theta,s)\Bigg]+\sum_{t=\theta}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}R_2^{'}(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_2[t,x]}{\partial a_2^2}$$

$$\times R_2(t+1,x;\,\theta,s)+\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}\Bigg[Q_3^{'}(t,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial z_3^2}\,Q_3(t,x;\,\theta,s)$$

$$+Q_3^{'}(t,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial z_3\,\partial a_3}Q_3(t+1,x;\,\theta,s)+Q_3^{'}(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial a_3\partial z_3}\,Q_3(t,x;\,\theta,s)$$

$$+Q_3^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3^2}\,Q_3(t+1,x;\,\theta,s)\Bigg], \tag{4.4}$$

$$K_3(\tau,s)=-R_3^{'}(t_3,X;\,\theta,\tau)\,\frac{\partial^2\varphi_3(z_3(t_3,X))}{\partial z_3^2}\,R_3(t_3,X;\,\theta,s)$$

$$+\sum_{t=\theta}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}\Bigg[R_3^{'}(t,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial z_3^2}\,R_3(t,x;\,\theta,s)$$

$$+R_3^{'}(t,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial z_3\,\partial a_3}\,R_3(t+1,x;\,\theta,s)+R_3^{'}(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial a_3\,\partial z_3}\,R_3(t,x;\,\theta,s)\Bigg]$$

$$+\sum_{t=\theta}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}R_3^{'}(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial a_3^2}\,R_3(t+1,x;\,\theta,s). \tag{4.5}$$

Using the discrete variants of line variations [30], we prove the following

Theorem 4.1 *If the sets U_i, $i=\overline{1,3}$ are open, then under the assumptions made for optimality of the classical extremal $u(t,x)$ in problem (2.1)-(2.4), (4.2) the following relations*

$$1)\qquad \sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1} v_1'(\tau)\,\frac{\partial f_1'[\theta,\tau]}{\partial u_1}\,K_1(\tau,s)\,\frac{\partial f_1[\theta,s]}{\partial u_1}\,v_1(s)+\sum_{x=x_0}^{x_1-1} v_1'(x)\,\frac{\partial^2 H_1[\theta,x]}{\partial u_1^2}\,v_1(x)$$
$$+2\sum_{x=x_0}^{X-1}\left[\sum_{s=x_0}^{x-1} v_1'(x)\,\frac{\partial^2 H_1[\theta,x]}{\partial u_1\,\partial a_1}\,R_1(\theta+1,x;\,\theta,s)\,\frac{\partial f_1[\theta,s]}{\partial u_1}\,v_1(s)\right]\le 0 \qquad (4.6)$$

should be fulfilled for all $v_1(x)\in R^r$, $x=x_0,\,x_0+1,\dots,X-1$, $\theta\in T_1=\{t_0,\,t_0+1,\,\dots,\,t_1-1\}$,

$$2)\qquad \sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1} v_2'(\tau)\,\frac{\partial f_2'[\theta,\tau]}{\partial u_2}\,K_2(\tau,s)\,\frac{\partial f_2[\theta,s]}{\partial u_2}\,v_2(s)+\sum_{x=x_0}^{x_1-1} v_2'(x)\,\frac{\partial^2 H_2[\theta,x]}{\partial u_2^2}\,v_2(x)$$
$$+2\sum_{x=x_0}^{X-1}\left[\sum_{s=x_0}^{x-1} v_2'(x)\,\frac{\partial^2 H_2[\theta,x]}{\partial u_2\,\partial a_2}\,R_2(\theta+1,x;\,\theta,s)\,\frac{\partial f_2[\theta,s]}{\partial u_2}\,v_2(s)\right]\le 0 \qquad (4.7)$$

for all $v_2(x)\in R^r$, $x=x_0,\,x_0+1,\dots,X-1$, $\theta\in T_2=\{t_1,\,t_1+1,\,\dots,\,t_2-1\}$,

$$3)\qquad \sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1} v_3'(\tau)\,\frac{\partial f_3'[\theta,\tau]}{\partial u_3}\,K_3(\tau,s)\,\frac{\partial f_3[\theta,s]}{\partial u_3}\,v_3(s)+\sum_{x=x_0}^{x_1-1} v_3'(x)\,\frac{\partial^2 H_3[\theta,x]}{\partial u_3^2}\,v_3(x)$$
$$+2\sum_{x=x_0}^{X-1}\left[\sum_{s=x_0}^{x-1} v_3'(x)\,\frac{\partial^2 H_3[\theta,x]}{\partial u_3\,\partial a_3}\,R_3(\theta+1,x;\,\theta,s)\,\frac{\partial f_3[\theta,s]}{\partial u_3}\,v_3(s)\right]\le 0 \qquad (4.8)$$

for all $v_3(x)\in R^r$, $x=x_0,\,x_0+1,\dots,X-1$, $\theta\in T_3=\{t_2,\,t_2+1,\,\dots,\,t_3-1\}$.

Proof. Using arbitrariness of admissible variations of the control $u(t,x)=(u_1(t,x),\,u_2(t,x),\,u_3(t,x))$, we assume

$$\delta u_1^*(t,x)=\begin{cases} v_1(x), & t=\theta\in T_1;\;\; x=x_0,\,x_0+1,\dots,\,X-1,\\ 0, & t\neq\theta;\;\; x=x_0,\,x_0+1,\dots,\,X-1,\end{cases} \qquad (4.9)$$

$$\delta u_1^*(t,x)=0,\quad (t,x)\in D_i,\quad i=1,2.$$

Here, $v_1(x)\in R^r$, $x=x_0,\,x_0+1,\dots,\,X-1$ is an arbitrary bounded vector-function, $\theta\in T_1=\{t_0,\,t_0+1,\,\dots,\,t_1-1\}$ is an arbitrary point.

By $\delta z^*(t,x)=(\delta z_1^*(t,x),\,\delta z_2^*(t,x),\,\delta z_3^*(t,x))$ we denote the solution of problems (3.5)-(3.6) that corresponds to special variation (4.9) of control. It follows from representations (3.7)-(3.9) that

$$\delta z_1^*(t,x)=\begin{cases} 0, & t=t_0,\,t_0+1,\dots,\,\theta;\;\; x=x_0,\,x_0+1,\dots,\,X,\\ \sum\limits_{s=x_0}^{x-1} R_1(t,x;\theta,s)\,\dfrac{\partial f_1[\theta,s]}{\partial u_1}v_1(s), & t\ge\theta+1;\;\; x=x_0,x_0+1,...,X,\end{cases} \qquad (4.10)$$

$$\begin{cases} \delta z_2^*(t,x) = \sum_{s=x_0}^{x-1} Q_1(t,x;\theta,s)\,\dfrac{\partial f_1[\theta,s]}{\partial u_1} v_1(s), \\ t = t_1, t_1+1, ..., t_2; \quad x = x_0, ..., X, \end{cases} \tag{4.11}$$

$$\begin{cases} \delta z_3^*(t,x) = \sum_{s=x_0}^{x-1} Q_2(t,x;\theta,s)\,\dfrac{\partial f_1[\theta,s]}{\partial u_1} v_1(s), \\ t = t_2, t_2+1, ..., t_3; \quad x = x_0, ..., X, \end{cases} \tag{4.12}$$

Allowing for (3.4), (4.5), (4.9) from (3.11) we get that for the optimality of classic singular control $u(t,x) = (u_1(t,x),\, u_2(t,x),\, u_3(t,x))$ in problem (2.1)-(2.4), (4.2) the inequality

$$\sum_{i=1}^{3} \delta z_i^{*'}(t_i,X) \frac{\partial^2 \varphi_i(z_i(t_i,X))}{\partial z_i^2} \delta z_i^*(t_i,X) - \frac{1}{2} \sum_{i=1}^{3} \Bigg[\sum_{t=t_{i-1}}^{t_i-1} \sum_{x=x_0}^{X-1} \Bigg[\delta z_i^{*'}(t,x) \frac{\partial^2 H_i[t,x]}{\partial z_i^2}$$
$$\times \delta z_i^*(t,x) + \delta z_i^{*'}(t,x) \frac{\partial^2 H_i[t,x]}{\partial z_i\,\partial a_i} \delta z_i^*(t+1,x) + \delta z_i^{*'}(t+1,x) \frac{\partial^2 H_i[t,x]}{\partial a_i\,\partial z_i} \delta z_i^*(t,x)$$
$$+\delta z_i^{*'}(t+1,x) \frac{\partial^2 H_i[t,x]}{\partial a_i^2} \delta z_i^*(t+1,x) \Bigg]\Bigg] - 2 \sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1-1} \Bigg[\delta u_1^{*'}(t,x) \frac{\partial^2 H_1[t,x]}{\partial u_1\,\partial z_1} \delta z_1^*(t,x)$$
$$+\delta u_1^{*'}(t,x) \frac{\partial^2 H_1[t,x]}{\partial u_1\,\partial a_1} \delta z_1^*(t+1,x) \Bigg] - \sum_{x=x_0}^{X-1} v_1'(x) \frac{\partial^2 H_1[t,x]}{\partial u_1^2} v_1(x) \geq 0, \tag{4.13}$$

should be fulfilled for all $v_1(x) \in R^r$, $x = x_0, x_0+1, ..., X-1$.

Further, using representations (4.10)-(4.12), we get

$$\sum_{i=1}^{3} \delta z_i^{*'}(t_i,X) \frac{\partial^2 \varphi_i(z_i(t_i,X))}{\partial z_i^2} \delta z_i^*(t_i,X) = \sum_{\tau=x_0}^{X-1} \sum_{s=x_0}^{X-1} v_1'(\tau) \frac{\partial f_1'[\theta,\tau]}{\partial u_1} \Bigg[R_1'(t_1,X;\,\theta,\tau)$$
$$\times \frac{\partial^2 \varphi_1(z_1(t_1,X))}{\partial z_1^2} R_1(t_1,X;\,\theta,s) + Q_1'(t_2,X;\,\theta,\tau)\,\frac{\partial^2 \varphi_2(z_2(t_2,X))}{\partial z_2^2} Q_1(t_2,X;\,\theta,s)$$
$$+Q_2'(t_3,X;\,\theta,\tau)\,\frac{\partial^2 \varphi_3(z_3(t_3,X))}{\partial z_3^2} Q_2(t_3,X;\,\theta,s) \Bigg]\,\frac{\partial f_1[\theta,s]}{\partial u_1} v_1(s). \tag{4.14}$$

By the scheme given in [25, 26], we have

$$\sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{X-1} \delta z_1^{*'}(t,x) \frac{\partial^2 H_1[t,x]}{\partial z_1^2} \delta z_1^*(t,x) = \sum_{\tau=x_0}^{X-1} \sum_{s=x_0}^{X-1} v_1'(\tau) \frac{\partial f_1'[\theta,\tau]}{\partial u_1}$$
$$\times \left[\sum_{t=\theta+1}^{t_1-1} \sum_{x=max(\tau,s)+1}^{X-1} R_1'(t,x;\,\theta,\tau) \frac{\partial^2 H_1[t,x]}{\partial z_1^2} R_1(t,x;\,\theta,s) \right]\,\frac{\partial f_1[\theta,s]}{\partial u_1} v_1(s),$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t,x)\frac{\partial^2 H_2[t,x]}{\partial z_2^2}\delta z_2^*(t,x)=\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=t_1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_1^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial z_2^2}Q_1(t,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t,x)\frac{\partial^2 H_3[t,x]}{\partial z_3^2}\delta z_3^*(t,x)=\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_2^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3^2}Q_2(t,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),\qquad(4.15)$$

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{X-1}\delta z_1^{*'}(t,x)\frac{\partial^2 H_1[t,x]}{\partial z_1\,\partial a_1}\delta z_1^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=\theta+1}^{t_1-1}\sum_{x=max(\tau,s)+1}^{X-1}R_1^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_1[t,x]}{\partial z_1\partial a_1}R_1(t+1,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{X-1}\delta z_1^{*'}(t+1,x)\frac{\partial^2 H_1[t,x]}{\partial a_1\,\partial z_1}\delta z_1^*(t,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=\theta+1}^{t_1-1}\sum_{x=max(\tau,s)+1}^{X-1}R_1^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_1[t,x]}{\partial a_1\partial z_1}R_1(t,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t,x)\frac{\partial^2 H_2[t,x]}{\partial z_2\,\partial a_2}\delta z_2^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=t_1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_1^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial z_2\partial a_2}Q_1(t+1,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t+1,x)\frac{\partial^2 H_2[t,x]}{\partial a_2\,\partial z_2}\delta z_2^*(t,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=t_1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_1^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial a_2\partial z_2}Q_1(t,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t,x)\frac{\partial^2 H_3[t,x]}{\partial z_3\,\partial a_3}\delta z_3^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$

$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_2^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3\partial a_3}Q_2(t+1,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t+1,x)\frac{\partial^2 H_3[t,x]}{\partial a_3\,\partial z_3}\delta z_3^*(t,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$

$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_2^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3\partial z_3}Q_2(t,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{X-1}\delta z_1^{*'}(t+1,x)\frac{\partial^2 H_1[t,x]}{\partial a_1^2}\delta z_1^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$

$$\times\left[\sum_{t=\theta}^{t_1-1}\sum_{x=max(\tau,s)+1}^{X-1}R_1^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_1[t,x]}{\partial a_1^2}R_1(t+1,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t+1,x)\frac{\partial^2 H_2[t,x]}{\partial a_2^2}\delta z_2^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$

$$\times\left[\sum_{t=t_1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_1^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial a_2^2}Q_2(t+1,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t+1,x)\frac{\partial^2 H_3[t,x]}{\partial a_3^2}\delta z_3^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$

$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_2^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3^2}Q_3(t+1,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s).$$

Further, on the basis of discrete analogy of Fubini theorem (see [20, 28, 29]), we get

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{X-1}\delta u_1^{*'}(t,x)\,\frac{\partial^2 H_1[t,x]}{\partial u_1\partial a_1}\,\delta z_1^*(t+1,x)=\sum_{x=x_0}^{X-1}\left[\sum_{s=x_0}^{X-1}v_1^{'}(x)\,\frac{\partial^2 H_1[\theta,x]}{\partial u_1\,\partial a_1}\,R_1(t+1,x;\,\theta,s)\right.$$
$$\left.\times\frac{\partial f_1[\theta,s]}{\partial u_1}\,v_1(s)\right]=\sum_{x=x_0}^{X-1}\left[\sum_{s=x+1}^{X-1}v_1^{'}(s)\,\frac{\partial^2 H_1[\theta,s]}{\partial u_1\,\partial a_1}\,R_1(t+1,s;\,\theta,x)\right]$$
$$\times\frac{\partial f_1[\theta,x]}{\partial u_1}\,v_1(x),\qquad(4.16)$$

Taking into account relations (4.14)-(4.16) and denotation (3.10) in relation (4.13) we arrive at inequality (4.6).

Now, we introduce the special variation of the control $u(t,x)$ by the formula

$$\begin{cases} \delta u_1^*(t,x) = 0, \quad (t,x) \in D_i, \quad i = 1,3, \\ \delta u_2^*(t,x) = \begin{cases} v_2(x), \quad t = \theta \in T_2; \;\; x = x_0,\, x_0+1, \ldots,\, X-1, \\ 0, \quad t \neq \theta; \;\; x = x_0,\, x_0+1, \ldots,\, X-1. \end{cases} \end{cases} \tag{4.17}$$

Here, $v_2(x) \in R^r$, $x = x_0,\, x_0+1, \ldots,\, X-1$ is an arbitrary r-dimensional bounded vector-function, $\theta \in T_2 = \{t_1,\, t_1+1,\, \ldots,\, t_2-1\}$ is an arbitrary point.

Denote by $\delta z^*(t,x) = (\delta z_1^*(t,x),\, \delta z_2^*(t,x),\, \delta z_3^*(t,x))$ the solution of problems (3.5)-(3.6) that corresponds to the special variation (35) of the control.

It follows from representations (3.7)-(3.9) that

$$\delta z_1^*(t,x) = 0,$$

$$\delta z_2^*(t,x) = \begin{cases} 0, \quad t = t_1,\, t_1+1, \ldots, \theta; \;\; x = x_0,\, x_0+1, \ldots,\, X, \\ \sum\limits_{s=x_0}^{x-1} R_2(t,x;\theta,s)\, \dfrac{\partial f_2[\theta,s]}{\partial u_2} v_2(s), \quad t \geq \theta+1, \end{cases} \tag{4.18}$$

$$\delta z_3^*(t,x) = \sum_{s=x_0}^{x-1} Q_3(t,x;\theta,s)\, \frac{\partial f_2[\theta,s]}{\partial u_2} v_2(s), \quad t = t_2, t_2+1, ..., t_3; \;\; x = x_0, ..., X.$$

Allowing for (3.4), (4.17), from (3.11) we obtain that for optimality of the classic extremal $u(t,x) = (u_1(t,x),\, u_2(t,x),\, u_3(t,x))$ in problem (1)-(4), (17) the inequality

$$\sum_{i=2}^{3} \delta z_i^{*'}(t_i,X) \frac{\partial^2 \varphi_i(z_i(t_i,X))}{\partial z_i^2} \delta z_i^*(t_i,X) - \frac{1}{2} \sum_{i=2}^{3} \Bigg[\sum_{t=t_{i-1}}^{t_i-1} \sum_{x=x_0}^{X-1} \Bigg[\delta z_i^{*'}(t,x) \frac{\partial^2 H_i[t,x]}{\partial z_i^2}$$
$$\times \delta z_i^*(t,x) + \delta z_i^{*'}(t,x) \frac{\partial^2 H_i[t,x]}{\partial z_i\, \partial a_i} \delta z_i^*(t+1,x) + \delta z_i^{*'}(t+1,x) \frac{\partial^2 H_i[t,x]}{\partial a_i\, \partial z_i} \delta z_i^*(t,x)$$
$$+ \delta z_i^{*'}(t+1,x) \frac{\partial^2 H_i[t,x]}{\partial a_i^2} \delta z_i^*(t+1,x) \Bigg] \Bigg] - 2 \sum_{t=t_1}^{t_2-1} \sum_{x=x_0}^{x_1-1} \Bigg[\delta u_2^{*'}(t,x) \frac{\partial^2 H_2[t,x]}{\partial u_2\, \partial z_2} \delta z_2^*(t,x)$$
$$+ \delta u_2^{*'}(t,x) \frac{\partial^2 H_2[t,x]}{\partial u_2\, \partial a_2} \delta z_2^*(t+1,x) \Bigg] - \sum_{x=x_0}^{X-1} v_2^{'}(x) \frac{\partial^2 H_2[t,x]}{\partial u_2^2} v_2(x) \geq 0, \tag{4.19}$$

should be fulfilled for all $v_2(x) \in R^r$, $x = x_0, x_0+1, ..., X-1$.

Further, using representations (4.18), we get

$$\sum_{i=1}^{3} \delta z_i^{*'}(t_i,X) \frac{\partial^2 \varphi_i(z_i(t_i,X))}{\partial z_i^2} \delta z_i^*(t_i,X) = \sum_{\tau=x_0}^{X-1} \sum_{s=x_0}^{X-1} v_2^{'}(\tau) \frac{\partial f_2^{'}[\theta,\tau]}{\partial u_2} \Bigg[R_2^{'}(t_2,X;\,\theta,\tau)$$
$$\times \frac{\partial^2 \varphi_2(z_2(t_2,X))}{\partial z_2^2} R_2(t_2,X;\,\theta,s) + Q_3^{'}(t_3,X;\,\theta,\tau)\, \frac{\partial^2 \varphi_3(z_3(t_3,X))}{\partial z_3^2} Q_3(t_3,X;\,\theta,s) \Bigg]$$
$$\times \frac{\partial f_2[\theta,s]}{\partial u_2} v_2(s). \tag{4.20}$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t,x)\frac{\partial^2 H_2[t,x]}{\partial z_2^2}\delta z_2^*(t,x)=\sum_{\tau=t_0}^{t_1-1}\sum_{s=x_0}^{X-1}v_2'(\tau)\frac{\partial f_2'[\theta,\tau]}{\partial u_2}$$

$$\times\left[\sum_{t=\theta+1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}R_2'(t,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial z_2^2}R_2(t,x;\,\theta,s)\right]\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s), \quad (4.21)$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t,x)\frac{\partial^2 H_3[t,x]}{\partial z_3^2}\delta z_3^*(t,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2'(\tau)\frac{\partial f_2'[\theta,\tau]}{\partial u_2}$$

$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_3'(t,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3^2}Q_3(t,x;\,\theta,s)\right]\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s), \quad (4.22)$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t,x)\frac{\partial^2 H_2[t,x]}{\partial z_2\,\partial a_2}\delta z_2^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2'(\tau)\frac{\partial f_2'[\theta,\tau]}{\partial u_2}$$

$$\times\left[\sum_{t=\theta+1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}R_2'(t,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial z_2\partial a_2}R_2(t+1,x;\,\theta,s)\right]\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s), \quad (4.23)$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t+1,x)\frac{\partial^2 H_2[t,x]}{\partial a_2\,\partial z_2}\delta z_2^*(t,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2'(\tau)\frac{\partial f_2'[\theta,\tau]}{\partial u_2}$$

$$\times\left[\sum_{t=\theta+1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}R_2'(t+1,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial a_2\partial z_2}R_2(t,x;\,\theta,s)\right]\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s), \quad (4.24)$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t,x)\frac{\partial^2 H_3[t,x]}{\partial z_3\,\partial a_3}\delta z_3^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2'(\tau)\frac{\partial f_2'[\theta,\tau]}{\partial u_2}$$

$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_3'(t,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3\partial a_3}Q_3(t+1,x;\,\theta,s)\right]\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s), \quad (4.25)$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t+1,x)\frac{\partial^2 H_3[t,x]}{\partial a_3\,\partial z_3}\delta z_3^*(t,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2'(\tau)\frac{\partial f_2'[\theta,\tau]}{\partial u_2}$$

$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_3'(t+1,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3\partial z_3}Q_3(t,x;\,\theta,s)\right]\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s), \quad (4.26)$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t+1,x)\frac{\partial^2 H_2[t,x]}{\partial a_2^2}\delta z_2^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2^{'}(\tau)\frac{\partial f_2^{'}[\theta,\tau]}{\partial u_2}$$
$$\times\left[\sum_{t=\theta}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}R_2^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial a_2^2}R_2(t+1,x;\,\theta,s)\right]$$
$$\times\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s), \qquad (4.27)$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t+1,x)\frac{\partial^2 H_3[t,x]}{\partial a_3^2}\delta z_3^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2^{'}(\tau)\frac{\partial f_2^{'}[\theta,\tau]}{\partial u_2}$$
$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_3^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3^2}Q_3(t+1,x;\,\theta,s)\right]$$
$$\times\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s). \qquad (4.28)$$

Using the discrete analogy of Foubini theorem [23], we have

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta u_2^{*'}(t,x)\frac{\partial^2 H_2[t,x]}{\partial u_2\,\partial a_2}\delta z_2^*(t+1,x)$$
$$=\sum_{x=x_0}^{X-1}\left[\sum_{s=x+1}^{X-1}v_2^{'}(s)\frac{\partial^2 H_2[\theta,s]}{\partial u_2\,\partial a_2}R_2(t+1,s;\,\theta,x)\right]\frac{\partial f_2[\theta,x]}{\partial u_2}v_2(x). \qquad (4.29)$$

Taking into account identities (4.20)-(4.29), and also denotation (4.4) in inequality (4.19), we arrive at relation (4.7). Inequality (4.8) is also proved by the appropriate arguments. This completes the proof of the theorem.

Remark. Similar symmetric results are obtained in the case when the right-hand side of system (2.1) has the form

$$f_i(t,x,z_i,a_i,b_i,u_i)=A_i(t,x)\,a_i+Q_i(t,x,z_i,b_i,u_i).$$

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