Seminar of I. Vekua Institute of Applied Mathematics REPORTS, Vol. 36-37, 2010-2011

ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF GENERALIZED EMDEN-FOWLER EQUATIONS WITH ADVANCED ARGUMENT

Koplatadze R., Kvinikadze G., Giorgadze G.

Abstract. The generalized Emden-Fowler Equation

$$u^{(n)}(t)+p(t)\big|u(\sigma(t))\big|^{\mu(t)}\operatorname{sign}u(\sigma(t))=0$$

is considered, where $p\in L_{\rm loc}(R_+;R_-)$, $\mu\in C(R_+;(0,+\infty))$, $\sigma\in C(R_+;R_+)$ and $\sigma(t)\geq t$ for $t\in R_+$. Oscillatory properties of solutions of the equation are studied. In particular, sufficient conditions are established for the equation to have Property **B**.

Keywords and phrases: Functional-differential equation, oscillation, Property B.

AMS subject classification (2000): 34K06; 34K11.

1. Introduction

In the paper the following differential equation is considered:

$$u^{(n)}(t)+p(t)\big|u(\sigma(t))\big|^{\mu(t)}\operatorname{sign}u(\sigma(t))=0, \tag{1.1}$$

where

$$\begin{gathered}p\in L_{\rm loc}(R_+;R_-),\quad \mu\in C(R_+;(0,+\infty)),\quad \sigma\in C(R_+;R_+)\\ \text{and}\quad \sigma(t)\geq t\quad\text{for}\quad t\in R_+.\end{gathered}\tag{1.2}$$

New sufficient conditions are established for oscillation of solutions of (1.1). Specifically, sufficient conditions are given for the equation (1.1) to have Property **B** (see below the definition of Property **B**).

A function $u:[t_0,+\infty)\to R$ is said to be a proper solution of (1.1), if it is locally absolutely continuous together with its derivatives up to the order $n-1$ inclusive, $\sup\{|u(s)|:s\geq t\}>0$ for $t\geq t_0$ and satisfies (1.1) almost everywhere on $[t_0,+\infty)$. A proper solution $u:[t_0,+\infty)\to R$ of the (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution u is said to be nonoscillatory.

Definition. We say that the equation (1.1) has Property **B** if any of its proper solutions either is oscillatory or satisfies

$$\big|u^{(i)}(t)\big|\downarrow 0\quad\text{as}\quad t\uparrow+\infty\quad(i=0,\dots,n-1)$$

or

$$\big|u^{(i)}(t)\big|\uparrow +\infty\quad\text{as}\quad t\uparrow+\infty\quad(i=0,\dots,n-1), \tag{1.3}$$

when n is even and either is oscillatory or satisfies (1.3) when n is odd.

In the present paper sufficient conditions of new type will be given for the equation (1.1) to have Property **B**. Analogous results for Property **A** are presented in [1]. As to almost linear equations (i.e. when $\lim\limits_{t\to+\infty}\mu(t)=1$), analogous issues for them are substantively studied in [2–4]. The result of the present paper make somewhat more complete those of [5] in case of Property **B**.

Let $t_0\in R_+$ and $\ell\in\{1,\dots,n-1\}$. By $\mathbf{U}_{\ell,t_0}$ we denote the set of all proper solutions $u:[t_0,+\infty)\to R$ of the equation (1.1) satisfying the conditions

$$\begin{aligned} u^{(i)}(t)>0 \quad &\text{for} \quad t\ge t_* \quad (i=0,\dots,\ell-1),\\ (-1)^{i+\ell}u^{(i)}(t)>0 \quad &\text{for} \quad t\ge t_* \quad (i=\ell,\dots,n-1), \end{aligned} \tag{1.4$_\ell$}$$

where $t_*\in[t_0,+\infty)$.

2. Sufficient conditions of nonexistence of solutions of the type (1.4_ℓ)

The assumption of the Theorems presented below contain one of the following two conditions:

$$\mu(t)\le\lambda<1 \quad \text{for} \quad t\in R_+ \tag{2.1}$$

or

$$\mu(t)\ge\lambda \quad \text{for} \quad t\in R_+ \quad \text{and} \quad \lambda\in(0,1). \tag{2.2}$$

The results of this section play an important role in establishing sufficient conditions for the equation (1.1) to have Property **B**.

Theorem 2.1. *Let the conditions* (1.2), (2.1) *and*

$$\int_0^{+\infty} t^{n-\ell-1}\big(\sigma(t)\big)^{\ell\mu(t)}|p(t)|dt=+\infty \tag{2.3$_\ell$}$$

be fulfilled and for some $\gamma\in(0,1)$

$$\liminf_{t\to+\infty}\ t^{\gamma}\int_t^{+\infty} s^{n-\ell-1+\mu(s)-\lambda}\big(\sigma(t)\big)^{(\ell-1)\mu(s)}|p(s)|ds>0, \tag{2.4$_\ell$}$$

where $\ell\in\{1,\dots,n-1\}$ *with* $\ell+n$ *even. If, moreover, for some* $\delta\in[0,\lambda]$ *and* $\sigma_*\in C(R_+)$ *such that*

$$t\le\sigma_*(t)\le\sigma(t) \quad \textit{for} \quad t\in R_+, \tag{2.5}$$

$$\int_0^{+\infty} t^{n-\ell-1+\lambda-\delta}\big(\sigma_*(t)\big)^{\mu(t)-\lambda+\frac{\delta(1-\gamma)}{1-\lambda}}\big(\sigma(t)\big)^{(\ell-1)\mu(t)}|p(t)|dt=+\infty,$$

then for any $t_0\in R_+$ *we have* $\mathbf{U}_{\ell,t_0}=\varnothing$.

Theorem 2.2 *Let the conditions* (1.2), (2.1), (2.3_ℓ) *and*

$$\liminf_{t\to+\infty}\ t\int_t^{+\infty} s^{n-\ell-1+\mu(s)-\lambda}\big(\sigma(s)\big)^{(\ell-1)\mu(s)}|p(s)|ds>0 \tag{2.6$_\ell$}$$

be fulfilled, where $\ell\in\{1,\dots,n-1\}$ *with* $\ell+n$ *even. If, moreover, for some* $\delta\in[0,\lambda]$ *and* $\sigma_*\in C(R_+)$ *satisfying the condition* (2.5) *the equality*

$$\int_0^{+\infty} t^{n-\ell-1+\lambda-\delta}\big(\sigma_*(t)\big)^{\mu(t)-\lambda}\big(\sigma(t)\big)^{\mu(t)(\ell-1)} \times\big(\ln(1+\sigma_*(t))\big)^{\frac{\delta}{1-\lambda}}|p(s)|ds=+\infty$$

holds, then for any $t_0 \in R_+$ *we have* $\mathbf{U}_{\ell,t_0} = \varnothing$.

Theorem 2.3. *Let the conditions* (1.2), (2.2), (2.3$_\ell$) *and*

$$\liminf_{t\to+\infty} t^\gamma \int_t^{+\infty} s^{n-\ell-1}\big(\sigma(s)\big)^{(\ell-1)\mu(s)}|p(s)|ds > 0 \tag{2.7$_\ell$}$$

be fulfilled, where $\gamma \in (0,1)$ *and* $\ell \in \{1,\dots,n-1\}$ *with* $\ell+n$ *even. If, moreover, for some* $\delta \in [0,\lambda]$ *the equality*

$$\int_0^{+\infty} t^{n-\ell-1+\delta}\big(\sigma(t)\big)^{(\ell-1)\mu(t)+\frac{(\mu(t)-\delta)(1-\gamma)}{1-\lambda}}|p(t)|dt = +\infty$$

holds, then for any $t_0 \in R_+$ *we have* $\mathbf{U}_{\ell,t_0} = \varnothing$.

Theorem 2.4. *Let the conditions* (1.2), (2.2) *and*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-\ell-1}\big(\sigma(s)\big)^{(\ell-1)\mu(s)}p(s)ds > 0 \tag{2.8$_\ell$}$$

be fulfilled, where $\ell \in \{1,\dots,n-1\}$ *with* $\ell+n$ *odd. If, moreover, for some* $\delta \in [0,\lambda]$ *the equality*

$$\int_0^{+\infty} t^{n-\ell-1+\lambda+\delta}\big(\sigma(t)\big)^{(\ell-1)\mu(t)}\big(\ln(1+\sigma(t)\big)^{\frac{\mu(t)-\delta}{1-\lambda}}|p(t)|dt = +\infty$$

holds, then for any $t_0 \in R_+$ *we have* $\mathbf{U}_{\ell,t_0} = \varnothing$.

3. Differential equations with property B (case $\mu(t) \le \lambda$)

Theorem 31. *Let the conditions* (1.2), (2.1), (2.3$_1$), (2.4$_1$) *and*

$$\liminf_{t\to+\infty} \frac{(\sigma(t))^{\mu(t)}}{t} > 0, \tag{3.1}$$

be fulfilled. If, moreover,

$$\int_0^{+\infty} t^{n-2+\mu(t)+\frac{\lambda(\lambda-\gamma)}{1-\lambda}}|p(t)|dt = +\infty,$$

then the equation (1.1) *has Property* **B**.

Theorem 3.2. *Let the conditions* (1.2), (2.1), (2.3$_1$), (2.4$_1$) *and* (3.1) *be fulfilled and*

$$\int_0^{+\infty} t^{n-2}\big(\sigma(t)\big)^{\mu(t)+\frac{\lambda(\lambda-\gamma)}{1-\lambda}}|p(t)|dt = +\infty.$$

Then the equation (1.1) *has Property* **B**.

Theorem 3.3. *Let the conditions* (1.2), (2.1), (2.3$_1$), (2.6$_1$) *and* (3.1) *be fulfilled and*

$$\int_0^{+\infty} t^{n-2+\mu(t)-\lambda}\big(\ln(1+t)\big)^{\frac{\lambda}{1-\lambda}}|p(t)|dt = +\infty.$$

Then the equation (1.1) *has Property* **B**.

Theorem 3.4. *Let the conditions* (1.2), (2.1), (3.1), (2.3_1) *and* (2.6_1) *be fulfilled and*

$$\int_0^{+\infty} t^{n-2}\big(\sigma(t)\big)^{\mu(t)-\lambda}\big(\ln(1+\sigma(t))\big)^{\frac{\lambda}{1-\lambda}}|p(t)|dt = +\infty.$$

Then the equation (1.1) *has Property* **B**.

Theorem 3.5. *Let the conditions* (1.2), (2.1), (2.3_{n-1}), (2.4_{n-2}) *and*

$$\limsup_{t\to+\infty} \frac{(\sigma(t))^{\mu(t)}}{t} < +\infty \tag{3.2}$$

be fulfilled. If, moreover,

$$\int_0^{+\infty} t^{\mu(t)+\frac{\lambda(\lambda-\gamma)}{1-\lambda}}\big(\sigma(t)\big)^{\mu(t)(n-3)}|p(t)|dt = +\infty,$$

then the equation (1.1) *has Property* **B**.

Theorem 3.6. *Let the conditions* (1.2), (2.1), (2.3_{n-1}), (3.4_{n-2}) *and* (3.2) *be fulfilled and*

$$\int_0^{+\infty} t\big(\sigma(t)\big)^{(n-2)\mu(t)+\frac{\lambda(1-\gamma)}{1-\lambda}}|p(t)|dt = +\infty.$$

Then the equation (1.1) *has Property* **B**.

Theorem 3.7 *Let the conditions* (1.2), (2.1), (2.3_{n-1}), (2.6_{n-2}) *and* (3.2) *be fulfilled and*

$$\int_0^{+\infty} t^{1+\mu(t)-\lambda}\big(\sigma(t)\big)^{(n-3)\mu(t)}\big(\ln(1+t)\big)^{\frac{\lambda}{1-\lambda}}|p(t)|dt = +\infty.$$

Then the equation (1.1) *has Property* **B**.

Theorem 3.8. *Let the conditions* (1.2), (2.1), (2.3_{n-2}), (2.6_{n-2}) *and* (3.2) *be fulfilled and*

$$\int_0^{+\infty} \big(\sigma(t)\big)^{(n-1)\mu(t)-\lambda}\big(\ln(1+\sigma(t))\big)^{\frac{\lambda}{1-\lambda}}|p(t)|dt = +\infty.$$

Then the equation (1.1) *has Property* **B**.

4. Differential equations with property B (case $\mu(t) \ge \lambda$)

Theorem 4.1. *Let the conditions* (1.2), (2.2), (2.3_1), (2.9_1) *and* (3.1) *be fulfilled and*

$$\int_0^{+\infty} t^{n-2}\big(\sigma(t)\big)^{\frac{\mu(t)(1-\gamma)}{1-\lambda}}|p(t)|dt = +\infty.$$

Then the equation (1.1) *has Property* **B**.

Theorem 4.2. *Let the conditions* (1.1), (1.2), (2.3_1), (2.9_1) *and* (3.1) *be fulfilled and*

$$\int_0^{+\infty} t^{1+\lambda}\big(\sigma(t)\big)^{\frac{(\mu(t)-\lambda)(1-\gamma)}{1-\lambda}}|p(t)|dt = +\infty.$$

Then the equation (1.1) *has Property* **B**.

Theorem 4.3. *Let the conditions* (1.2), (2.2), (3.1), (2.3_1) *and* (2.8_1) *be fulfilled and*

$$\int_0^{+\infty} t^{n-2}\big(\ln(1+\sigma(t))\big)^{\frac{\mu(t)}{1-\lambda}}|p(t)|dt=+\infty.$$

Then the equation (1.1) *has Property* **B**.

Theorem 4.4. *Let the conditions* (1.2), (2.2), (2.3_1), (2.8_1) *and* (3.1) *be fulfilled and*

$$\int_0^{+\infty} t^{n-2+\lambda}\big(\ln(1+\sigma(t))\big)^{\frac{\mu(t)-\lambda}{1-\lambda}}|p(t)|dt=+\infty.$$

Then the equation (1.1) *has Property* **B**.

Theorem 4.5. *Let the conditions* (1.2), (2.2), (2.3_{n-1}), (2.7_{n-2}) *and* (3.2) *be fulfilled and*

$$\int_0^{+\infty} t\big(\sigma(t)\big)^{(n-3)\mu(t)+\frac{\mu(t)(1-\gamma)}{1-\lambda}}|p(t)|dt=+\infty.$$

Then the equation (1.1) *has Property* **B**.

Theorem 4.6. *Let the conditions* (1.2), (2.2), (2.3_{n-1}), (2.7_{n-2}) *and* (3.2) *be fulfilled and*

$$\int_0^{+\infty} t^{1+\lambda}\big(\sigma(t)\big)^{(n-3)\mu(t)+\frac{(\mu(t)-\lambda)(1-\gamma)}{1-\lambda}}|p(t)|dt=+\infty.$$

Then the equation (1.1) *has Property* **B**.

Theorem 4.7. *Let the conditions* (1.2), (2.2), (2.3_{n-1}), (3.2) *and* (2.8_{n-2}), *be fulfilled and*

$$\int_0^{+\infty} t\big(\sigma(t)\big)^{(n-3)\mu(t)}\big(\ln(1+\sigma(t))\big)^{\frac{\mu(t)}{1-\lambda}}|p(t)|dt=+\infty.$$

Then the equation (1.1) *has Property* **B**.

Theorem 4.8. *Let the conditions* (1.2), (2.2), (2.3_{n-1}) *and* (2.8_{n-2}) *be fulfilled and*

$$\int_0^{+\infty} t^{1+\lambda}\big(\sigma(t)\big)^{(n-3)\mu(t)}\big(\ln(1+\sigma(t))\big)^{\frac{\mu(t)-\lambda}{1-\lambda}}|p(t)|dt=+\infty.$$

Then the equation (1.1) *has Property* **B**.

Acknowledgement. The work was supported by the Sh. Rustaveli National Science Foundation (Georgia). Grant No. GNSF/ST09-81-3-101.

REFERENCES

1. Koplatadze R., Kvinikadze G., Giorgadze G. Oscillatory properties of solution of generalized Emden-Fowler equations with advenced argument. *Proc. Razmadze Math. Inst.*, 2010 (accepted).

2. Koplatadze R., Litsyn E. Oscillation criteria for higher order "almost linear" functional differential equations. *Funct. Differ. Equ.*, **16**, 3 (2009), 387-434.

3. Koplatadze R. On asymptotic behaviors of solutions of "almost linear" and essential nonlinear functional differential equations. *Nonlinear Anal.* **71**, 12 (2009), 396-400.

4. Koplatadze R. Quasi-linear functional differential equations with property A. *J. Math. Anal. Appl.*, **330**, 1 (2007), 483-510.

5. Koplatadze R., Kvinikadze G. Nonlinear generalized equations of Emden-Fowler type with advanced argument. *Math. Modeling and Computer Simulation of Matherial Technologies. Proccedings of the* 6*-th International Conference Ariel*, 2010 (accepted).

Received 27.04.2010; revised 13.06.2010; accepted 17.07.2010.

Authors' addresses:

R. Koplatadze
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: $r\_$ koplatadze@yahoo.com

G. Kvinikadze
National Statistics Office of Georgia
4, Pekini St., Tbilisi, 0115
Georgia
E-mail: gkvinikadze@statistics.gov.ge

G. Giorgadze
Department of Mathematics
Georgian Technical University
77, M. Kostava St., Tbilisi 0175
Georgia
E-mail: g_ givi@hotmail.com