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ON THE UNIQUENESS OF THE SOLUTION OF AN INVERSE PROBLEM OF THE POTENTIALLY THEORY IN A THREE-DIMENSIONAL SPACE

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Abstract. In the present paper we consider the inverse problem for a volume potential. First we consider piecewise-smooth simply-connected domains and after that smooth simply-connected domains in a three-dimensional space.

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The solution of an inverse problem of the potential theory is of great theoretical and practical importance. The practical application of inverse problems is so significant that they are regarded as topical problems of modern mathematical analysis.

The uniqueness of the solution of an inverse problem in the class of star domains of constant density was for the first time proved P.S. Novikov [1].

In the present paper we consider the inverse problem for a volume potential.First we consider piecewise-smooth simply-connected domains and after that smooth simply-connected domains in a three-dimensional space.

Let us define volume potentials and simple-layer potentials.

$$V^f(x) = \int\limits_{\Omega} \Gamma(x,y) f(y) dS_y, \quad U^{\psi}(x) = \int\limits_{\partial\Omega} \Gamma(x,y)\psi(y) dS_y,$$

where Ω is a bounded piecewise-smooth domain, $f \in C(\partial\Omega)$, $\psi \in C(\partial\Omega)$, $\Gamma(x,y) = |x-y|^{-1}$. We denote by Ω_∞ the simply-connected component of $R^3 - \overline{\Omega}$ which contains a point at infinity, and by $\emptyset$ an empty set. C_k, $k = 1, 2, 3, \ldots$ are positive constants.

Definition 1. Let Q be a simply-connected bounded piecewise-smooth domain from R^3. We will set the domain Q is strictly convex if for any points $z_1 \in \overline{Q}$, $z_2 \in \overline{Q}$ an interval point of a segment $\overline{z_1 z_2}$ is an interval point for the domain Q.

Definition 2. Let Ω be a simply-connected bounded piecewise-smooth domain from R^3, and each smooth part for $\partial\Omega$ belongs to class $C^{(1,\alpha)}$. We will say that the domain Ω is strictly convex at a point $x_0 \in \partial\Omega$ if for some neighborhood $\sigma = \{x : |x - x_0| < \varepsilon\}$ the intersection $\overline{\Omega} \cap \overline{\sigma}$ is a strict domain.

Theorem 1. *Let Ω_1, Ω_2 be a bounded simple-connected domain from R^3. Assume that there exists a smooth point $x_0 \in \partial\Omega_1$, $x_0 \notin \overline{\Omega}_2$, for which the domain Ω_1 or $R^3 - \overline{\Omega}_1$ is strictly convex at a point x_0. Then the potentials*

$$v_1(x) = \int\limits_{\Omega_1} \Gamma(x,y) dy, \quad v_2(x) = \int\limits_{\Omega_2} \Gamma(x,y) dy \tag{1}$$

do not coincide on Ω_∞ ($\Omega = \Omega_1 \cup \Omega_2$, $\partial\Omega_i = \partial\overline{\Omega}_i$, $i = 1, 2$.)

Proof. Let us assume the contrary,i.e. $v_1(x) = v_2(x)$, $x \in \Omega_\infty$. Each smooth part $\partial\Omega_i$ $(i = 1, 2)$ belongs to $C^{(1,\alpha)}$. Denote $\sigma_1 = \{x : |x - x_0| < \varepsilon\} \cap \partial\Omega_1$, $x_0 = (x_1^0, x_2^0)$, $\sigma = \left\{x : |x - x_0| < \dfrac{\varepsilon}{2}\right\} \cap \partial\Omega_1$, $(\overline{\sigma}_1 \cap \overline{\Omega}_2 = \emptyset)$, $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \subset S = \{x : |x| < r\}$. Rotate the coordinate system. After rotation the tangent plane at the point x_0 becomes parallel to the plane x_2ox_3. Pass the plan P_1 through the point x_0, for which $P_1 \perp x_2ox_3$, $P_1 \perp x_10x_2$. Let us consider the curve $l = P_1 \cap \sigma$, the equation of which has the form $x_3 = \tau(x_1, c_1)$, $x_2 = c_1 = const$, $x \in l$, $x_1^0 - \varepsilon_1 < x_1 < x_1^0$, $|\tau'(x_1^0, c)| = \infty$.

From equality (1) we obtain

$$\int\limits_{\Omega_1} U^\psi(x)dx = \int\limits_{\Omega_2} U^\psi(x)dx, \quad \psi \in C^{(1,\alpha)}(\partial S)$$

$$\int\limits_{\Omega_1} \frac{\partial U^\psi}{\partial x_3}dx = \int\limits_{\Omega_2} \frac{\partial U^\psi}{\partial x_3}dx \quad \psi \in C^{(1,\alpha)}(\partial S) \tag{2}$$

By the Green-Ostrogradski formula, from (2) we have

$$\int\limits_{\Omega_1} U^\psi(x)\cos(\nu_x^\wedge x_3)dS_x = \int\limits_{\Omega_2} U^\psi(x)\cos(\nu_x^\wedge x_3)dS_x \tag{3}$$

Let ω be the domain containing the surface σ.

For any $\psi \in C(\sigma)$ the following inequality is valid

$$||\psi||_{\{C^3(\sigma)^*\}} \le C_1 ||U^\psi||_{\{C_0^1(\omega)\}^*}, \tag{4}$$

where $C_0^1(\omega)$ - are finite functions from $C^1(\overline{\omega})$. It is obvious that the boundary function on σ'

$$g(x_1, x_2) \cdot x_3 = g(x_1x_2) \cdot \tau(x_1x_2)$$

$x_3 = \tau(x_1x_2)$ is the equation $\sigma' = \{(x_1x_2x_3) \in \sigma,\ x_3 < x_3^0\}$, $L_1 = \dfrac{\partial \delta_{x_1}}{\partial t} \cdot \delta_{x_2}$, $(x_1, x_2, x_3) \in \sigma'$, where $\delta_{x_1}, \delta_{x_2}$ are Dirac measures.

From this and the above reasoning we obtain for a ball $\left\{||U^\psi||_{\{C_0^1(\omega)\}^*} \le 1\right\}$ $\left(||\psi||_{\{C^3(\sigma)\}^*} \le C_1\right)$.

$$\frac{1}{C_1}\frac{\partial \delta_{x_1}}{\partial t} \times \delta_{x_2} \cdot x_3 \in \left\{||U^\psi||_{\{C_0^1(\omega)\}} \le 1\right\},\ U^{\psi_1}(x_1, x_2, x_3) = \frac{1}{C_1}\frac{\partial \delta_{x_1}}{\partial t} \times \delta_{x_2} \cdot x_3,\ (x_1x_2x_3) \in \sigma'.$$

$$\sup \left| \int\limits_{\sigma_1} U^\psi(x_1, x_2, x_3)\cos(\nu_x \hat{}\, x_3)dS_x \right| \ge C_2 \sup \left| \int\limits_{\sigma'} U^\psi(x_1, x_2)\tau(x_1x_2^0)dx_1dx_2 \right| = \infty. \tag{5}$$

By virtue of (4) we have

$$\int\limits_{\sigma_1} U^\psi(x)\cos(\nu_x^\wedge x_3)dS_x = \int\limits_{\partial\Omega_2} U^\psi(x)\cos(\nu_x^\wedge x_3)dS_x - \int\limits_{\partial\Omega_1 - \sigma_1} U^\psi(x)\cos(\nu_x^\wedge x_3)dS_x \tag{6}$$

$$\sup_{\partial\Omega_2} |U^\psi(x)| \le C_3, \quad \sup_{\partial\Omega_1-\sigma_1} |U^\psi(x)| \le C_4, \quad \psi \in \{C^3(\sigma)\}^*. \tag{7}$$

From (5), (6), (7) we obtain a contradiction.

Theorem 1 is proved.

Theorem 2. *Let Ω_1 and Ω_2 be simply connected bounded domains from the class C^2. Then the solution of an inverse problem is unique.*

Proof. Let us assume the contrary, i.e. that $v_1(x) = v_2(x)$, $x \in \Omega_\infty$, $\Omega_1 \neq \Omega_2$.

For the domains Ω_1 and Ω_2 the following alternatives are valid.

I) $\partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega_\infty$ - is a finite number of smooth curves.

II) $\partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega_\infty$ - contains some smooth surface σ.

Assume that alternative (I) is fulfilled. Consider the diameter of the domain $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$

$$d(\overline{\Omega}) = \max |x-y|, \quad x \in \overline{\Omega}, \ y \in \overline{\Omega}. \ \ d(\overline{\Omega}) = |x_0 - y_0|.$$

It is not difficult to see that in the neighborhood of point x_0 (or y_0) there exists a smooth point $z_0 \in \partial\overline{\Omega}_1$, $z_0 \notin \overline{\Omega}_2$, for which the domain Ω_1 is strictly locally convex at a point z_0. Now is suffices to repeat the reasoning of Theorem 1.

Assume that alternative (II) is fulfilled. Consider the difference

$$(\overline{\Omega}_1 \cup \overline{\Omega}_2) - (\overline{\Omega}_1 \cap \overline{\Omega}_2) = \bigcup_1^N Q_i.$$

Since $\sigma \subset \partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega_\infty$, the complement of the closed set $F = \bigcup_1^N \overline{Q}_i$ is a connected set (domain). Now assume that the potentials

$$v_1(x) = \int_{\Omega_1} \Gamma(x,y)dy, \quad v_2(x) = \int_{\Omega_2} \Gamma(x,y)dy$$

are considered on Ω_∞. Then we obtain

$$\int_{\overline{\Omega}_1-(\overline{\Omega}_1\cap\overline{\Omega}_2)} U^\psi(y)dy = \int_{\overline{\Omega}_2-(\overline{\Omega}_1\cap\overline{\Omega}_2)} U^\psi(y)dy \quad \psi \in C(\partial\Omega_\infty). \tag{8}$$

By virtue of Keldish theorem [2, ch. II] there exists a sequence of potentials for which we obtain

$$\lim_{n\to\infty} \int_{Q_1} [U^{\psi_n}(x) - 1]^2 dx = 0, \quad \lim_{n\to\infty} \int_{F-Q_1} [U^{\psi_n}(x)]^2 dx = 0.$$

$$\int_{Q_1} dy = 0, \quad |Q_1| = 0.$$

We have come to a contradiction. Theorem 2 is proved.

R E F E R E N C E S

1. Novikov P.S. On the uniqueness of the solution of an inverse problem of the potentially theory. (Russian) *Proc. of the Academy of Sciences of USSR*, **18** 3 (1938).

2. Keldish M.B. Mathematics. Selected Workers. (Russian) *Nauka, Moscow*, 1985.

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