Seminar of I. Vekua Institute
of Applied Mathematics
REPORTS, Vol. 36-37, 2010-2011

A REMARK CONCERNING PECULIARITIES OF TWO MODELS OF CUSPED PRISMATIC SHELLS

Jaiani G.

Abstract. Comparative analysis of peculiarities of setting of boundary value problems are carried out for cusped prismatic shells within the framework of the zero approximation of hierarchical models when on the face surfaces either stress or displacement vectors are assumed to be known.

Keywords and phrases: Cusped plates, cusped prismatic shells, mathematical modeling, linear elasticity, degenerate and singular elliptic and hyperbolic equations and systems.

AMS subject classification (2000): 74K20; 74K25; 74B05; 35Q74; 35J70; 35J75; 35L80; 35L81.

Let $Ox_1x_2x_3$ be an anticlockwise-oriented rectangular Cartesian frame of origin O. We conditionally assume the x_3-axis vertical. The elastic body is called a prismatic shell if it is bounded above and below by, respectively, the surfaces (so called face surfaces)

$$x_3 = \overset{(+)}{h}(x_1, x_2) \text{ and } x_3 = \overset{(-)}{h}(x_1, x_2),$$

laterally by a cylindrical surface Γ of generatrix parallel to the x_3-axis and its vertical dimension is sufficiently small compared with other dimensions of the body.

In other words, the 3D elastic prismatic shell-like body occupies a bounded region $\overline{\Omega}$ with boundary $\partial\Omega$, which is defined as:

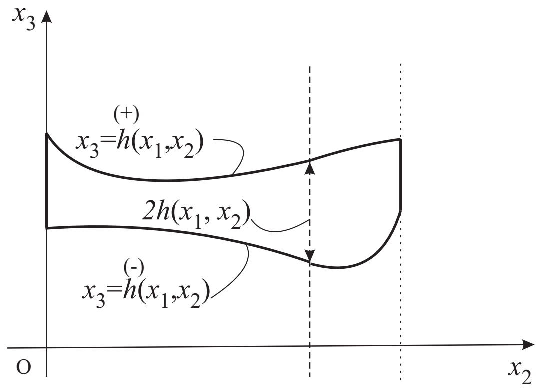

Fig.1. A cross-section of a typical non-cusped prismatic shell

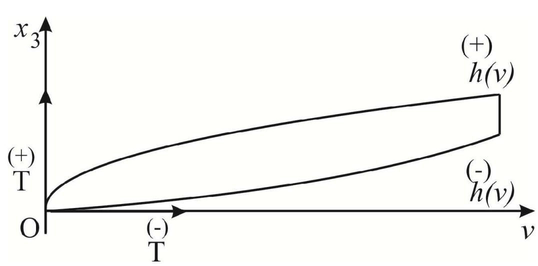

Fig.2. A cross-section of a blunt cusped prismatic shell

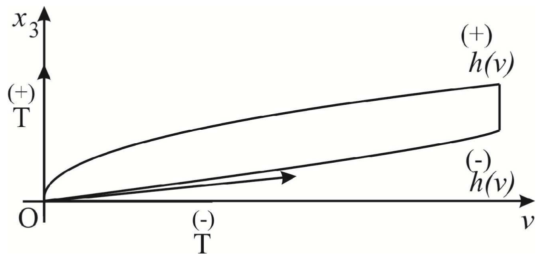

Fig.3. A cross-section of a blunt cusped prismatic shell ($\varphi \in]0, \frac{\pi}{2}[$)

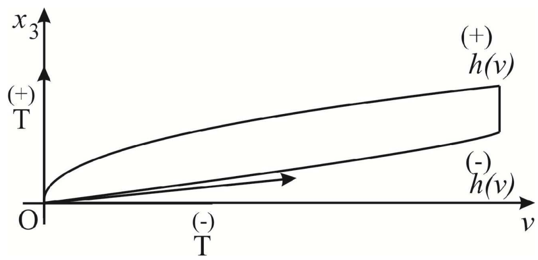

Fig.4. A cross-section of a blunt cusped prismatic shell ($\varphi = 0$)

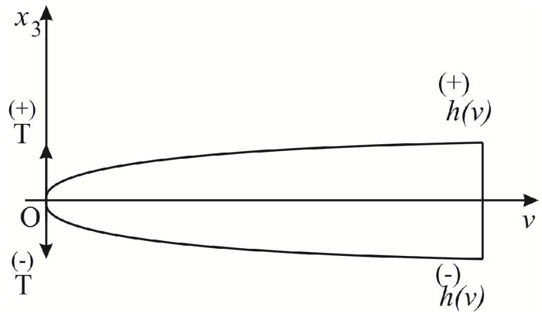

Fig.5. A cross-section of a blunt cusped plate ($\varphi = \pi$)

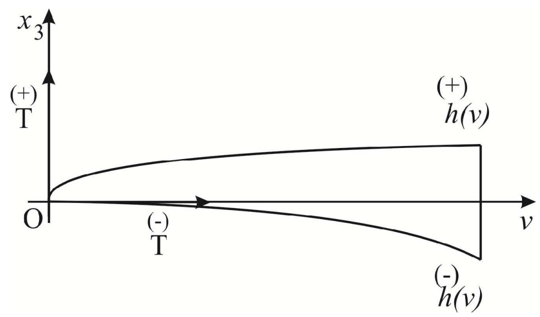

Fig.6. A cross-section of a blunt cusped prismatic shell ($\varphi = \frac{\pi}{2}$)

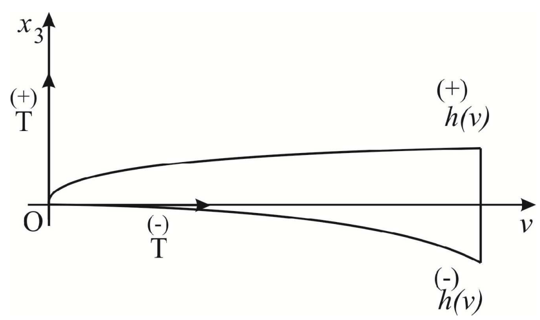

Fig.7. A cross-section of a blunt cusped prismatic shell ($\varphi \in]\frac{\pi}{2}, \pi[$)

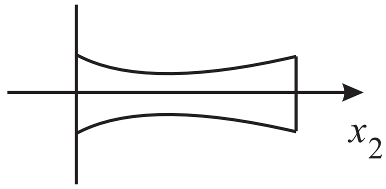

Fig.8. Non-cusped edges

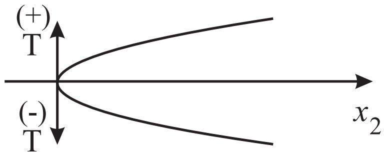

Fig.9. $\varphi = \pi$

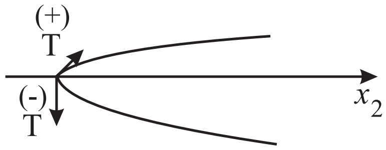

Fig.10. $\frac{\pi}{2} < \varphi < \pi$

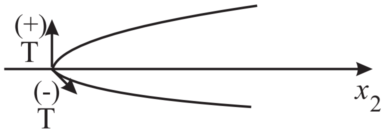

Fig.11. $\frac{\pi}{2} < \varphi < \pi$

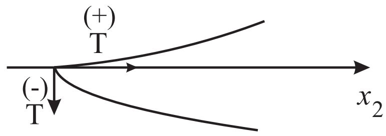

Fig.12. $\varphi = \frac{\pi}{2}$

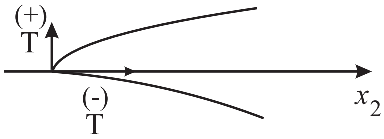

Fig.13. $\varphi = \frac{\pi}{2}$

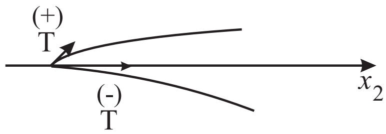

Fig.14. $0 < \varphi < \frac{\pi}{2}$

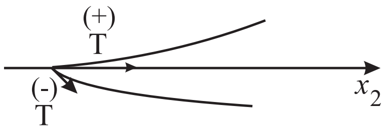

Fig.15. $0 < \varphi < \frac{\pi}{2}$

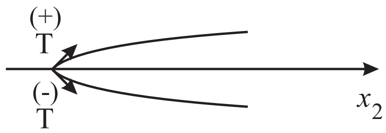

Fig.16. $0 < \varphi < \pi$

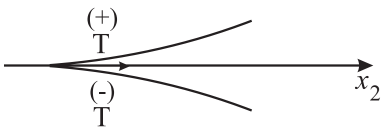

Fig.17. $\varphi = 0$

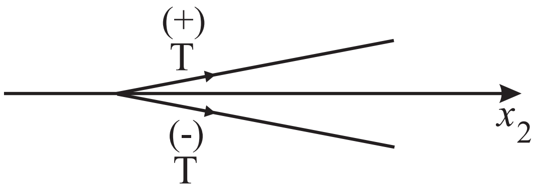

Fig.18. Wedge

Typical cross-sections of prismatic shells

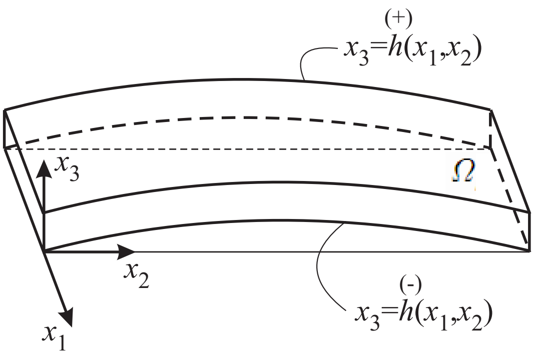

Fig.19. Prismatic shell of constant thickness

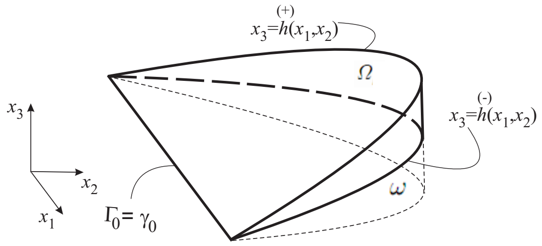

Fig.20. A sharp cusped prismatic shell with a semicircle projection

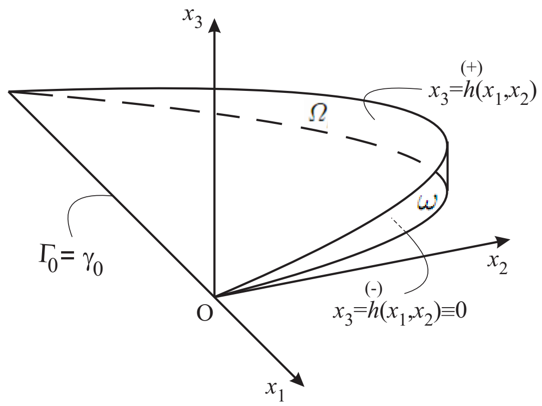

Fig.21. A sharp cusped prismatic shell with a semicircle projection

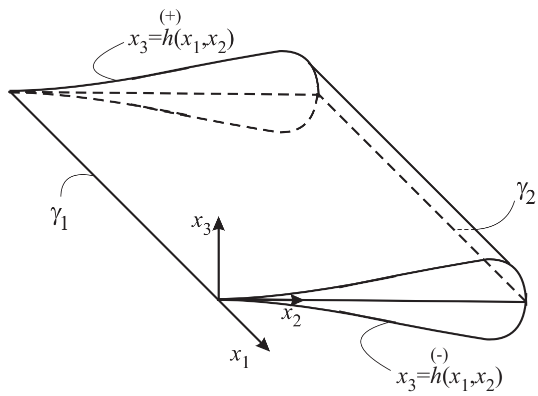

Fig.22. A cusped plate with sharp γ_1 and blunt γ_2 edges, $\gamma_0 = \gamma_1 \cup \gamma_2$

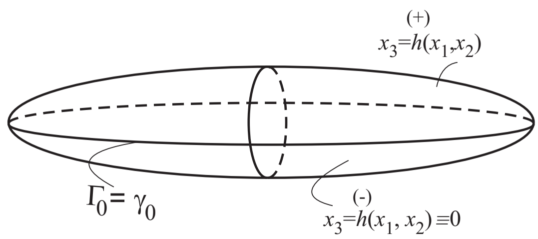

Fig.23. A blunt cusped plate with the edge γ_0

$$\Omega := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \; (x_1, x_2) \in \omega, \; \overset{(-)}{h}(x_1, x_2) < x_3 < \overset{(+)}{h}(x_1, x_2) \right\},$$

where $\overline{\omega} := \omega \cup \partial\omega$ is the so-called projection of the prismatic shell $\overline{\Omega} := \Omega \cup \partial\Omega$ (see Figures 1-18, where typical cross-sections of prismatic shells with an angle φ between tangents $\overset{(+)}{T}$ and $\overset{(-)}{T}$ are given and Figures 19-23); $\gamma = \partial\omega$ and $\partial\Omega$ denote boundaries of ω and Ω, respectively; $\mathbb{R}^n$ is an n-dimensional Euclidian space.

In what follows we assume that

$$\overset{(\pm)}{h}(x_1, x_2) \in C^2(\omega) \cap C(\bar{\omega}),^{1}$$

and

$$2h(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2) \begin{cases} > 0 & \text{for} \quad (x_1, x_2) \in \omega, \\ \geq 0 & \text{for} \quad (x_1, x_2) \in \partial\omega \end{cases}$$

is the thickness of the prismatic shell $\overline{\Omega}$ at the points $(x_1, x_2) \in \bar{\omega} = \omega \cup \partial\omega$. $\max\{2h\}$ is essentially less than characteristic dimensions of ω. Let

$$2\overline{h}(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) + \overset{(-)}{h}(x_1, x_2).$$

In the symmetric case of the prismatic shells, i.e., when

$$\overset{(-)}{h}(x_1, x_2) = -\overset{(+)}{h}(x_1, x_2), \quad \text{i.e.,} \quad 2\overline{h}(x_1, x_2) = 0,$$

we have to do with plates of variable thickness $2h(x_1, x_2)$ and a middle-plane ω (see Figures 22, 23). Prismatic shells are called cusped ones if a set γ_0, consisting of $(x_1, x_2) \in \partial\omega$ for which $2h(x_1, x_2) = 0$, is not empty. For such prismatic shells $\partial\Omega$ may be non-Lipschitz boundary (see Fig. 22)

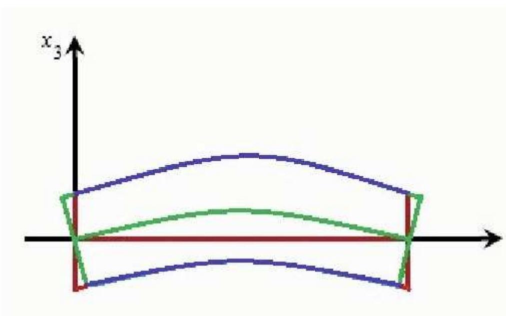

Fig.24. Comparison of cross-sections of prismatic and standard shells

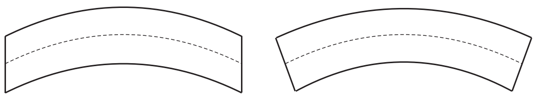

Fig.25. Cross-sections of a prismatic (left) and a standard shell with the same mid-surface

Distinctions between the prismatic shell of constant thickness and the standard shell of constant thickness are shown on Figures 24 and 25. The lateral boundary of the standard shell is orthogonal to the middle surface of the shell, while the lateral

[1] $C(\bar{\omega})$ denotes a class of continuous on $\bar{\omega}$ functions; $C^2(\omega)$ denotes a class of twice continuously dofferentiable functions with respect to x_1, x_2, $(x_1, x_2) \in \omega$.

boundary of the prismatic shell is orthogonal to the projection of the prismatic shell on $x_3 = 0$.

In what follows X_{ij} and e_{ij} are the stress and strain tensors, respectively, u_i are the displacements, Φ_i are the volume force components, ρ is the density, λ and μ are the Lamé constants, δ_{ij} is the Kroneker delta, subscripts preceded by a comma mean partial derivatives with respect to the corresponding variables. Moreover, repeated indices imply summation (Greek letters run from 1 to 2, and Latin letters run from 1 to 3, unless stated otherwise).

I.Vekua's hierarchical models for elastic prismatic shells are the mathematical models, which were introduced by I. Vekua [1, 2], and which were constructed by the multiplication of the basic equations of linear elasticity

Motion Equations

$$X_{ij,j} + \Phi_i = \rho \ddot{u}_i(x_1, x_2, x_3, t), \quad (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3, \quad t > t_0, \quad i = 1, 2, 3;$$

Generalized Hooke's law (isotropic case)

$$X_{ij} = \lambda \theta \delta_{ij} + 2\mu e_{ij}, \quad i, j = 1, 2, 3, \quad \theta := e_{ii};$$

Kinematic Relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 2, 3,$$

by Legendre polynomials $P_l(ax_3 - b)$, $l = 0, 1, 2, \dots$, where

$$a(x_1, x_2) := \frac{1}{h(x_1, x_2)}, \quad b(x_1, x_2) := \frac{\overline{h}(x_1, x_2)}{h(x_1, x_2)},$$

and then integration with respect to x_3 within the limits $\overset{(-)}{h}(x_1, x_2)$ and $\overset{(+)}{h}(x_1, x_2)$. By these calculations in Vekua's first version on upper and lower face surfaces stress-vectors are assumed as prescribed, while values of the displacements are calculated there from their (displacements') Fourier-Legendre series expansions on the segment $x_3 \in \left[\overset{(-)}{h}(x_1, x_2), \overset{(+)}{h}(x_1, x_2)\right]$ and vice versa in his second version. So, we get the equivalent infinite system of relations with respect to the so called l-th order moments

$$\begin{aligned}\Big(X_{ijl},\, e_{ijl},\, u_{il}\Big)(x_1, x_2, t) \quad := & \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} \Big(X_{ij},\, e_{ij},\, u_i\Big)(x_1, x_2, x_3, t) \\ \times & \quad P_l(ax_3 - b)\; dx_3. \end{aligned} \tag{1}$$

Then, having followed the usual procedure used in the theory of elasticity, we get an equivalent infinite system with respect to the l-th order moments u_{il}. After this if we assume that the moments whose subscripts, indicating order of moments are greater than N equal zero and consider only the first $N + 1$ equations (for every $i = 1, 2, 3$)

in the obtained infinite system of equations with respect to the l-th order moments u_{il} we obtain the $N-$th order approximation (hierarchical model) governing system with respect to $\overset{N}{u}_{il}$ (roughly speaking $\overset{N}{u}_{il}$ is an "approximate value" of u_{il}).

In the zero approximation of I.Vekua's hierarchical models of shallow prismatic shells the governing system has the form

$$\mu\big[(hv_{\alpha 0,\beta})_{,\alpha}+(hv_{\beta 0,\alpha})_{,\alpha}\big]+\lambda(hv_{\gamma 0,\gamma})_{,\beta}=-\overset{0}{X}_{\beta}+\rho h\ddot{v}_{\beta 0},\quad \beta=1,2, \tag{2}$$

$$\mu(hv_{30,\alpha})_{,\alpha}=-\overset{0}{X}_{3}+\rho h\ddot{v}_{30}, \tag{3}$$

where $v_{k0}:=\frac{u_{k0}}{h}$, $k=1,2,3$, are unknown so called weighted "moments" of displacements,

$$\begin{aligned}\overset{0}{X}_{j}&:=\overset{(+)}{\sigma}_{3j}-\overset{(+)}{\sigma}_{\alpha j}\overset{(+)}{h}_{,\alpha}+(-1)^{r}\Big[-\overset{(-)}{\sigma}_{3j}+\overset{(-)}{\sigma}_{\alpha j}\overset{(-)}{h}_{,\alpha}\Big]+\Phi_{j0}\\ &=Q_{\overset{(+)}{n}j}\sqrt{1+\Big(\overset{(+)}{h}_{,1}\Big)^{2}+\Big(\overset{(+)}{h}_{,2}\Big)^{2}}\\ &+(-1)^{r}Q_{\overset{(-)}{n}j}\sqrt{1+\Big(\overset{(-)}{h}_{,1}\Big)^{2}+\Big(\overset{(-)}{h}_{,2}\Big)^{2}}+\Phi_{j0},\quad j=1,2,3,\quad r=\overline{0,N}.\end{aligned}$$

By $Q_{\overset{(+)}{n}j}$ and $Q_{\overset{(-)}{n}j}$ components of the stress vectors acting on the upper and lower surfaces, respectively, are denoted. By Φ_{j0} we denote the zero order moments of the components of the volume forces.

When on the face surfaces displacements are prescribed for $N=0$ approximation the governing system has the following form

$$\begin{aligned}&\mu\left[(hv_{\alpha 0})_{,\beta}+(hv_{\beta 0})_{,\alpha}\right]_{,\beta}+\lambda\left[(hv_{\gamma 0})_{,\gamma}\right]_{,\alpha}\\ &-(\ln h)_{,\beta}\left\{\lambda\delta_{\alpha\beta}(hv_{\gamma 0})_{,\gamma}+\mu\left[(hv_{\alpha 0})_{,\beta}+(hv_{\beta 0})_{,\alpha}\right]\right\}\\ &+2\mu\,\Psi_{\alpha\beta,\beta}(x_1,x_2,t)+\lambda\,\Psi_{kk,\alpha}(x_1,x_2,t)\\ &-(\ln h)_{,\beta}\left[\lambda\delta_{\alpha\beta}\Psi_{kk}(x_1,x_2,t)+2\mu\,\Psi_{\alpha\beta}(x_1,x_2,t)\right]\\ &+\Phi_{\alpha 0}(x_1,x_2,t)=\rho h\ddot{v}_{\alpha 0},\quad \alpha=1,2;\end{aligned} \tag{4}$$

$$\begin{aligned}&\mu(hv_{30})_{,\beta\beta}-(\ln h)_{,\beta}\,\mu(hv_{30})_{,\beta}+2\mu\,\Psi_{3\beta,\beta}(x_1,x_2,t)\\ &-2\mu(\ln h)_{,\beta}\,\Psi_{3\beta}(x_1,x_2,t)+\Phi_{30}(x_1,x_2,t)=\rho h\ddot{v}_{30},\end{aligned} \tag{5}$$

where

$$\Psi_{33}(x_1,x_2,t):=u_3(x_1,x_2,\overset{(+)}{h},t)-u_3(x_1,x_2,\overset{(-)}{h},t),$$

$$2\Psi_{i\beta}(x_1,x_2,t):=u_i\left(x_1,x_2,\overset{(-)}{h},t\right)\overset{(-)}{h}_{,\beta}-u_i\left(x_1,x_2,\overset{(+)}{h},t\right)\overset{(+)}{h}_{,\beta}$$

$$+\begin{cases}-u_\beta\left(x_1,x_2,\overset{(+)}{h},t\right)\overset{(+)}{h}_{,\alpha}+u_\beta\left(x_1,x_2,\overset{(-)}{h},t\right)\overset{(-)}{h}_{,\alpha} & \text{for } i=\alpha,\quad \alpha=1,2;\\ u_\beta\left(x_1,x_2,\overset{(+)}{h},t\right)-u_\beta\left(x_1,x_2,\overset{(-)}{h},t\right) & \text{for } i=3.\end{cases}$$

Let now
$$2h = h_0 x_2^\kappa, \quad h_0,\, \kappa = const > 0, \ \ x_2 \geq 0. \tag{6}$$
In the static case, for deflections from (3) we get
$$\mu(hv_{30,\alpha})_{,\alpha} = -\overset{0}{X}_3, \ \ x_2 \geq 0.$$
Assuming that u_{30} depends only on x_2 (i.e., we consider cylindrical deformation)
$$(x_2^\kappa v_{30,\alpha})_{,\alpha} = -2\mu^{-1}h_0^{-1}\overset{0}{X}_3,$$
whence,
$$v_{30,22} + \frac{\kappa}{x_2}v_{30,2} = -2\mu^{-1}h_0^{-1}x_2^{-\kappa}\overset{0}{X}_3, \tag{7}$$
The general solution of the latter has the form
$$v_{30} = 2(\kappa-1)^{-1}\mu^{-1}h_0^{-1}\int\limits_{x_2^0}^{x_2}\left(x_2^{1-\kappa} - \xi^{1-\kappa}\right)\overset{0}{X}_3(\xi)d\xi \tag{8}$$
$$+c_1x_2^{1-\kappa} + c_2, \ \ \kappa \neq 1, \ \ c_1,\, c_2 = const;$$
$$v_{30} = 2\mu^{-1}h_0^{-1}\int\limits_{x_2^0}^{x_2}(\ln\xi - \ln x_2)\overset{0}{X}_3(\xi)d\xi + c_1\ln x_2 + c_2, \tag{9}$$
$$\kappa = 1, \ \ x_2^0 \in]0, l[, \ \ c_1,\, c_2 = const.$$

Hence, under the evident assumption on $\overset{0}{X}_3$, it is easy to conclude that on the boundary $x_2 = 0$ in the class of bounded functions displacement $\frac{v_{30}}{2}$ can be prescribed when $0 \leq \kappa < 1$, while for $\kappa \geq 1$ the boundary $x_2 = 0$ should be freed from the boundary condition (BC). Boundary value problems (BVPs) and initial boundary value problems (IBVPs) for the system (2), (3) and in the general N-th approximation are studied sufficiently well in the case of cusped prismatic shells (see [3-18]). For prismatic cusped shells the system (4), (5) is not studied at all. If we consider the case (6) for equation (5), it is easy to see that the systems (2), (3) and (4), (5) qualitatively differ from each other.

In the static case, from (5) we get
$$\begin{aligned}&\mu(hv_{30})_{,\beta\beta} - (\ln h)_{,\beta}\,\mu(hv_{30})_{,\beta} + 2\mu\ \Psi_{3\beta,\beta}(x_1,x_2)\\ &-2\mu(\ln h)_{,\beta}\,\Psi_{3\beta}(x_1,x_2) + \Phi_{30}(x_1,x_2) = 0,\end{aligned} \tag{10}$$

i.e.,
$$\begin{aligned}&hv_{30,\beta\beta} + 2h_{,\beta}v_{30,\beta} + h_{,\beta\beta}v_{30} - (\ln h)_{,\beta}(hv_{30,\beta} + h_{,\beta}v_{30})\\ &= -2\Psi_{3\beta,\beta} + 2(\ln h)_{,\beta}\,\Psi_{3\beta} - \mu^{-1}\Phi_{30}.\end{aligned}$$

Therefore,
$$\begin{aligned}&hv_{30,\beta\beta} + h_{,\beta}v_{30,\beta} + [h_{,\beta\beta} - (\ln h)_{,\beta}h_{,\beta}]v_{30}\\ &= -2\Psi_{3\beta,\beta} + 2(\ln h)_{,\beta}\,\Psi_{3\beta} - \mu^{-1}\Phi_{30}.\end{aligned} \tag{11}$$

Assuming that $\Phi_{30} \in C(\bar{\omega})$, $u_\alpha \equiv 0$, $\alpha = 1, 2$, and v_{30} depends only on x_2, taking into account (6) and dividing the equality (11) on $\frac{h_0}{2} x_2^{\kappa-2}$, from (11) we get

$$x_2^2 v_{30,22} + \kappa x_2 \nu_{30,2} - \kappa v_{30} = 2h_0^{-1}[-2x_2^{2-\kappa}\Psi_{32,2} + 2\kappa x_2^{1-\kappa}\Psi_{32} - \mu^{-1} x_2^{2-\kappa}\Phi_{30}]. \qquad (12)$$

The last equation is well-known Euler equation and, since $\kappa+1 > 0$, its general solution has the form

$$\begin{aligned} v_{30} = \frac{u_{30}}{\frac{h_0}{2} x_2^\kappa} = -2(\kappa+1)^{-1} h_0^{-1} \int\limits_{x_2^0}^{x_2} (x_2 \xi^{-\kappa} - x_2^{-\kappa}\xi) \qquad (13) \\ \times \Big[2\Psi_{32,2}(\xi) - 2\kappa\xi^{-1}\Psi_{32}(\xi) + \mu^{-1}\Phi_{30} \Big] d\xi \\ +2h_0^{-1} c_1 x_2 + 2h_0^{-1} c_2 x_2^{-\kappa}, \quad 0 < x_2^0 < L, \end{aligned}$$

where c_1 and c_2 are arbitrary constants.

The last results can also be achieved as follows: if we rewrite (5) with respect to u_{30}

$$\mu(u_{30}),_{\beta\beta} - (\ln h),_\beta \, \mu(u_{30}),_\beta = -2\mu\Psi_{3\beta,\beta} + 2\mu(\ln h),_\beta \, \Psi_{3\beta} - \Phi_{30}$$

and take into account (6) we get

$$u_{30,22} - \frac{\kappa}{x_2} u_{30,2} = -2\Psi_{32,2} + 2\frac{\kappa}{x_2}\Psi_{32} - \mu^{-1}\Phi_{30}. \qquad (14)$$

Its general solution has the form

$$u_{30} = -(\kappa+1)^{-1} \int\limits_{x_2^0}^{x_2} (x_2^{1+\kappa}\xi^{-\kappa} - \xi)\Psi(\xi) d\xi + c_1 x_2^{1+\kappa} + c_2, \qquad (15)$$

where

$$\Psi(\xi) := 2\Psi_{32,2}(\xi) - \frac{2\kappa}{\xi}\Psi_{32}(\xi) + \mu^{-1}\Phi_{30}(\xi).$$

Hence, since in the zero approximation it is assumed that

$$u_i(x_1, x_2, x_3, t) = \frac{1}{2h} u_{i0}(x_1, x_2, t) =: \frac{1}{2} v_{i0}(x_1, x_2, t),$$

we obtain (13).

Note that, in view of (15),

$$\begin{aligned} X_{320}(x_2) &= \mu(h v_{30}),_2 + 2\mu\Psi_{32}(x_2) = \mu u_{30,2} + 2\mu\Psi_{32}(x_2) \\ &= \mu c_1(\kappa+1) x_2^\kappa - \mu x_2^\kappa \int\limits_{x_2^0}^{x_2} \xi^{-\kappa}\Psi(\xi) d\xi + 2\mu\Psi_{32}(x_2). \end{aligned}$$

Clearly, if $\overset{(+)}{h}(x_2) = h_1 x_2^\kappa,\ \overset{(-)}{h}(x_2) = h_2 x_2^\kappa,\ h_1, h_2 = const,\ h_1 > h_2\ (h_0 := h_1 - h_2)$,

$$\lim_{x_2\to 0} X_{320}(x_2) = \frac{\mu}{\kappa}\lim_{x_2\to 0}\Big(2x_2\Psi_{32,2} - 2\kappa\Psi_{32} + \mu^{-1}x_2\Phi_{30}\Big) + 2\mu\lim_{x_2\to 0}\Psi_{32}$$

$$= \frac{2\mu}{\kappa}\lim_{x_2\to 0} x_2\Psi_{32,2}$$

$$= \frac{2\mu}{\kappa}\begin{cases} 0\ \text{ if }\ \kappa > 1\ \text{ and }\ u_3;\ u_{3,2} = O(1),\ \ x_2 \to 0; \\ \kappa(\kappa-1)(\overset{(-)}{d}_1 h_2 - \overset{(+)}{d}_1 h_1)\ \text{ if }\ 0 < \kappa \le 1\ \text{ and }\ u_{3,2} = O(1),\ \ x_2 \to 0, \\ \quad u_3(x_1, x_2, \overset{(\pm)}{h}(x_2)) = \overset{(\pm)}{\psi}(x_1, x_2)x_2^{1-\kappa},\ \ \lim\limits_{x_2\to 0}\overset{(\pm)}{\psi}(x_1, x_2) = \overset{(\pm)}{d}_1; \\ O^*(x_2^{\kappa-1}) = d_0\kappa(\kappa-1)x_2^{\kappa-1},\ \ x \to 0,\ \text{ if } 0 < \kappa < 1\ \text{ and }\ u_{3,2} = O(1), \\ \quad \lim\limits_{x_2\to 0} u_3(x_1, x_2, \overset{(\pm)}{h}(x_2)) = d_0 \ne 0. \end{cases}$$

Since under assumption of boundedness of 3D u_3, all its moments (because of boundedness of the integrand in (1) and tending of integration limits to 0 as $x_2 \to 0$) vanish at cusped edge, in particular

$$u_{30}(0) = 0$$

should be fulfilled. It will be achieved if in (15) we take

$$c_2 = -(\kappa+1)^{-1}\int\limits_{x_2^0}^{0}\xi\Big[2\Psi_{32,2}(\xi) - 2\kappa\xi^{-1}\Psi_{32}(\xi) + \mu^{-1}\Phi_{30}(\xi)\Big]d\xi, \tag{16}$$

This is easily seen because of

$$\lim_{x_2\to 0} x_2^{\kappa+1}\int\limits_{x_2^0}^{x_2}\xi^{-\kappa}\Big[2\Psi_{32,2}(\xi) - 2\kappa\xi^{-1}\Psi_{32}(\xi) + \mu^{-1}\Phi_{30}(\xi)\Big]d\xi = 0.$$

If (16) is violated, then, by virtue of (15), taking into account the last limit, $u_{30}(0) \ne 0$ and from (13) it follows that v_{30} is unbounded as $x_2 \to 0$, which contradicts the boundedness of u_3.

Applying the general representation (13) of v_{30}, let us analyze the setting of bending BVPs on $[0, L]$.

If c_2 has the form (16), then, by virtue of (13), (15),

$$\lim_{x_2\to 0} v_{30} = \lim_{x_2\to 0}\frac{u_{30}}{\frac{h_0}{2}x_2^\kappa} = \lim_{x_2\to 0}\frac{2\Big\{c_2 - (\kappa+1)^{-1}\int\limits_{x_2^0}^{x_2}(x_2^{\kappa+1}\xi^{-\kappa} - \xi)\Psi(\xi)d\xi\Big\}}{h_0 x_2^\kappa}$$

$$= \lim_{x_2\to 0}\frac{-2(\kappa+1)^{-1}(x_2^{\kappa+1}x_2^{-\kappa} - x_2)\Psi(x_2) - x_2^\kappa\int\limits_{x_2^0}^{x_2}\xi^{-\kappa}\Psi(\xi)d\xi}{\kappa h_0 x_2^{\kappa-1}}$$

$$= \lim_{x_2 \to 0} \left[\frac{0}{\kappa h_0 x_2^{\kappa-1}} - \frac{x_2}{\kappa h_0} \int\limits_{x_2^0}^{x_2} \xi^{-\kappa} \Psi(\xi) d\xi \right].$$

Therefore,

$$\lim_{x_2 \to 0} v_{30}(x_2) = 0 - \frac{1}{\kappa h_0} \lim_{x_2 \to 0} x_2 \int\limits_{x_2^0}^{x_2} \xi^{-\kappa} \Psi(\xi) d\xi \tag{17}$$

if Ψ is such a function that there exists the last limit.

Thus,

$$\begin{aligned} v_{30}(x_2) &= 2h_0^{-1} c_1 x_2 + 2h_0^{-1} (\kappa+1)^{-1} x_2^{-\kappa} \\ &\times \left\{ \int\limits_0^{x_2} \xi \Psi(\xi) d\xi - x_2^{\kappa+1} \int_{x_2^0}^{x_2} \xi^{-\kappa} \Psi(\xi) d\xi \right\} \end{aligned} \tag{18}$$

is bounded near $x_2 = 0$ under some restrictions on Ψ and choosing appropriately c_1 we can satisfy either BC

$$v_{30}(L) = v_{30}^L \tag{19}$$

or BC

$$X_{320}(L) = \mu (h v_{30})_{,2}|_{x_2 = L} + 2\mu \Psi_{32}(L) = \mu u_{30,2}|_{x_2=L} + 2\mu \Psi_{32}(L) = X_{320}^L. \tag{20}$$

Namely, correspondingly,

$$c_1 = 2^{-1} h_0 L^{-1} v_{30}^L - (\kappa+1)^{-1} \left\{ L^{-\kappa-1} \int\limits_0^L \xi \Psi(\xi) d\xi - \int\limits_{x_2^0}^L \xi^{-\kappa} \Psi(\xi) d\xi \right\} \tag{21}$$

and

$$c_1 = (1+\kappa)^{-1} \mu^{-1} L^{-\kappa} X_{320}^L + (1+\kappa)^{-1} \int\limits_{x_2^0}^L \xi^{-\kappa} \Psi(\xi) d\xi - 2(1+\kappa)^{-1} L^{-\kappa} \Psi_{32}(L). \tag{22}$$

Under some restrictions on Ψ from boundedness of u_3 there follows boundedness of $v_{30} \in C^2(]0, L[) \cap C(]0, L])$, which given by (18) with (21) is a unique solution of the BVP (12), (19), when $\kappa > 0$. Thus, actually we have solved the Keldysh type BVP.

If volume forces and the displacement on the face surfaces are equal to zero, i.e., $\Phi_{30} \equiv 0$, $\Psi_{32} \equiv 0$, it is natural to set BC on the edge $x_2 = 0$ as

$$v_{30}(0) = 0 \tag{23}$$

since the last follows from (17).

(18) with (21) gives a unique solution of BVP $(12)_0$[2], (23), (19), of the form

$$v_{30}(x_2) = \frac{\nu_{30}^L}{L} x_2.$$

[2] $(12)_0$ means homogeneous equation (12).

This BVP is not correct since by inhomogeneous BC (23) it will not be solvable. In order to get correct BVP, BC (23) should be replaced by boundedness of the solution, so, we again arrive at the correct Keldysh type BVP.

As it follows from the general representation (8), (9) of the solution v_{30} of equation (7) analogous BVP for equation (7) (the model, when stress vectors on the face surfaces are prescribed) is uniquely solvable only if $0 \leq \kappa < 1$, moreover, the non-homogenous BC (23) is admissible in contrast to the previous model (see (12)). When $\kappa \geq 1$ under condition of boundedness of v_{30} it is possible to satisfy only one BC.

Remark. In the case under consideration under assumption of boundedness of 3D displacements it follows from (14), (15) that

$$u_{30,22} - \frac{\kappa}{x_2} u_{30} = 0, \tag{24}$$

$$u_{30} = c_1 x_2^{1+\kappa} + c_2.$$

Evidently, BVP (24),

$$u_{30}(0) = u_{30}^0, \quad u_{30}(L) = u_{30}^L,$$

is uniquely solvable provided that u_{30}^0 and u_{30}^L are assumed to be known. From 3D BVP in displacements u_{30}^L is known, while $u_{30}^0 = 0$ and cannot be arbitrarily prescribed. If nevertheless we find u_{30}^0 to be assigned, displacement v_{30} will become unbounded as $x_2 \to 0$, which will be nonsense since ∞ cannot be approximate value of 0. While zero can be considered as approximate boundary value since we consider small deflections. In such sense we could consider (23) as BC when $\Psi_{32} \not\equiv 0$.

Now, let us analyze the possibility of prescribing the stress vectors on the prismatic shell edges.

Since

$$X_{320}(x_2) = \mu u_{30,2} = \frac{1}{2} \mu h_0 (x_2^{\kappa} v_{30})_{,2}\,,$$

by virtue of (15),

$$X_{320}(x_2) = \mu(1+\kappa) c_1 x_2^{\kappa}.$$

The last means that

$$X_{320}(0) = 0.$$

Hence, X_{320} can be arbitrarily prescribed only at non-cusped edge $x_2 = L$.

For the homogeneous equation $(12)_0$ besides the BC (23) we can set the BC (20), i.e., on the edge $x_2 = L$ the stress vector is given.

(18) with (22) gives a unique solution of BVP $(12)_0$, (23), (20) of the form

$$v_{30} = \frac{2X_{320}^L}{\mu h_0 (\kappa+1) L^{\kappa}} x_2.$$

Considering (8) we easily conclude that analogous BVP $(7)_0$,(23),(20), is uniquely solvable for the model (7), provided that $0 \leq \kappa < 1$ (in this case also the non-homogenous BC (19) is admissible). For $\kappa \geq 1$ from (8), (9) it is easily seen that only bounded solution is a constant and if $X_{320}^L \neq 0$, BVP $(7)_0$, (23), (20), is not solvable. If $X^L_{320} = 0$, then a solution of BVP $(7)_0$, nonhomogeneous (23), $(20)_0$ is a constant given at $x_2 = 0$.

Conclusion. In the case of the first model [see (7)] the Dirichlet problem is correct for $0 < \kappa < 1$ and the Keldysh problem is correct for $\kappa \geq 1$, while in the case of the second model [see (12)] the Keldysh problem is correct for $\kappa > 0$.

R E F E R E N C E S

1. Vekua I.N. On one method of calculating of prismatic shells. (Russian) *Trudy Tbilis. Mat. Inst.*, **21** (1955), 191-259.

2. Vekua I.N. Shell theory: General Methods of Construction. Pitman Advanced Publishing Program, *Boston-London-Melbourne*, 1985.

3. Jaiani G.V. On a wedge-shaped body arbitrary loaded along the cusped edge. (Russian) *Bull. Georgian Acad. Sci.*, **75**, 2 (1974), 309-312.

4. Jaiani G.V. On a physical interpretation of Fichera's function. *Acad. Naz. dei Lincei, Rend. della Sc. Fis. Mat. e Nat.*, S. **8**, 68 (1980), fasc. 5, 426-435.

5. Jaiani G.V. Solution of Some Problems for a Degenerate Elliptic Equation of Higher Order and Their Applications to Prismatic Shells. (Russian) *Tbilisi University Press*, 1982.

6. Jaiani, G.V. Boundary value problems of mathematical theory of prismatic shells with cusps. (Russian) *Proceedings of All-union Seminar in Theory and Numerical Methods in Shell and Plate Theory, Tbilisi University Press*, 1984 .

7. Jaiani G.V. The first boundary value problem of cusped prismatic shell theory in zero approximation of Vekua's theory. (Russian) *Proc. I. Vekua Inst. Appl. Math.*, **29** (1988), 5-38.

8. Jaiani G.V. Elastic bodies with non-smooth boundaries–cusped plates and shells. *ZAMM Z. Angew. Math. Mech.*, **76**, 2 (1996), 117-120.

9. Jaiani G.V. Application of Vekua's dimension reduction method to cusped plates and bars. *Bull. TICMI*, **5** (2001), 27-34 (for electronic version see: http://www.viam.science.tsu.ge/others/ticmi).

10. Jaiani G.V. A cusped prismatic shell-like body with the angular projection under the action of a concentrated force. *Rendiconti Academia Nazionale delle Scienze detta dei XL, Memorie di Matematica e Applicazioni*, 124^0, XXX, facs. **1** (2006), 65-82.

11. Jaiani G.V. A cusped prismatic shell-like Body under the action of concentrated moments. *Z. Angew. Math. Phys.*, **59** (2008), 518-536.

12. Jaiani G. On physical and mathematical moments and the setting of boundary conditions for cusped prismatic shells and beams. *Proceedings of the IUTAM Symposium on Relation of Shell, plate, Beam, and 3D Models Dedicated to Centenary of Ilia Vekua,* 23-27 April, 2007, *Tbilisi, Georgia. IUTAM Bookseries,* **9** (2008), 133-146, *Editors G. Jaiani, P. Podio-Guidugli, Springer.*

13. Jaiani G., Kharibegashvili S., Natroshvili D., Wendland W.L. Hierarchical models for elastic cusped plates and beams. *Lect. Notes TICMI*, **4** (2003).

14. Jaiani G., Kharibegashvili S., Natroshvili D., Wendland W.L. Two-dimensional hierarchical models for prismatic shells with thickness vanishing at the boundary. *J. Elasticity*, **77**, 2 (2004), 95-122.

15. Jaiani G., Schulze B.W. Some degenerate elliptic systems and applications to cusped plates. *Mathematische Nachrichten*, **280**, 4 (2007), 388-407.

16. Chinchaladze N. Vibration of an elastic plate under the action of an incompressible fluid. *Proceedings of the IUTAM Symposium on Relation of Shell, plate, Beam, and 3D Models Dedicated to Centenary of Ilia Vekua,* 23-27 April, 2007, *Tbilisi, Georgia. IUTAM Bookseries,* **9** (2008), 77-90, *Editors G. Jaiani, P. Podio-Guidugli, Springer.*

17. Chinchaladze N. Vibration of an elastic plate under action of an incompressible fluid in case of N=0 approximation of I. Vekua's hierarchical models. *Appl. Anal.*, **85**, 9 (2006), 1177-1187.

18. Chinchaladze N., Jaiani G., Maistrenko B., Podio-Guidugli P. Concentrated contact interactions in cuspidate prismatic shell-like bodies. *Archive of Applied Mechanics*, **81**, 10 (2011), 1487-1505.

Received 30.04.2011; revised 25.11.2011; accepted 5.12.2011.

Author's address:

G. Jaiani
I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: george.jaiani@gmail.com