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# SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS IN THIN PRISMATIC DOMAINS

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**Abstract**. The paper is devoted to a dimension reduction method for solving boundary value and initial boundary value problems of systems of partial differential equations in thin non-Lipschitz, in general, prismatic domains.

**Keywords and phrases**: Partial differential equations, order degeneration, dimension reduction method, thin non-Lipschitz prismatic domains.

**AMS subject classification (2000):** 35A25; 35J70; 35J75; 35K65; 35K67; 35L80; 35L81.

The paper deals with the system of $n$ first order linear partial differential equations

$$A_{ijk}u_{j,k} + B_{ij}u_j + C_i(x) = 0, \quad i = 1, 2, \dots, n, \tag{1}$$

where

$$A_{ijk}, B_{ij} = const \ \text{ and functions } \ C_i(x), \quad i, j = 1, 2, \dots, n, \quad k = 1, 2, 3, \ \text{are given}$$

(under repeated index $j$ the sum from 1 to $n$ is meant, under repeated $k$ the sum from 1 to 3 is meant, and under repeated Greek indices the sum from 1 to 2 is meant), in $n$ unknown functions $u_i(x_1, x_2, x_3)$ of three variables in the following non-Lipschitz, in general, 3D prismatic domain with the Lipschitz 2D projection $\omega$ on $x_3 = 0$:

$$\Omega := \left\{ x := (x_1, x_2, x_3) \in R^3 : (x_1, x_2) \in \omega, \quad \overset{(-)}{h}(x_1, x_2) < x_3 < \overset{(+)}{h}(x_1, x_2) \right\}$$

where $2h := \overset{(+)}{h} - \overset{(-)}{h} > 0$ in $\omega \cup \gamma_1$, $2h = 0$ on $\gamma_0$; $\partial\omega = \bar{\gamma}_0 \cup \bar{\gamma}_1$, $\nu$ is an inward normal to $\partial\omega$. Each of $\gamma_0$ and $\gamma_1$ may be empty but , clearly, not at the same time. When $\dfrac{\partial h}{\partial \nu} = 0$ on $\gamma_0$, the domain $\Omega$ is a non-Lipschitz one.

The boundary value problems for the system (1) in the $3D$ non-Lipschitz, in general, domain $\Omega$ can be reduced to the boundary value problems in the Lipschitz $2D$ domain $\omega$ for the infinite system of singular first order partial differential equations with respect to the s. c. weighted Legendre moments (see [1,2]) of the unknown functions $u_i(x_1, x_2, x_3)$:

$$v_{ir}(x_1, x_2) = \frac{u_{ir}(x_1, x_2)}{h^{r+1}}, \quad i = 1, 2, ..., n, \quad r = 0, 1, ..., \tag{2}$$

where

$$u_{ir}(x_1, x_2) = \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} u_i(x_1, x_2, x_3)\, P_r(ax_3 - b)\, dx_3,$$

$$a = \frac{1}{h}, \quad b = \frac{\tilde{h}}{h}, \quad 2\tilde{h} = \overset{(+)}{h} + \overset{(-)}{h}.$$

By this approach difficulties caused by the geometrical singularity of the 3D domain are reduced to the singularity of the equations. In other words, we avoid consideration of 3D non-Lipschitz domains but we get the infinite system of partial differential equations with singular coefficients in 2D Lipschitz domains. In order to present this we apply I.Vekua's dimension reduction method [1,2]. To this end we multiply both the sides of the system (1) by $P_r(ax_3-b)$ and the obtained expressions integrate within the limits $\overset{(-)}{h}(x_1,x_2)$ and $\overset{(+)}{h}(x_1,x_2)$:

$$
\begin{aligned}
&A_{ij\alpha}\Bigg[u_{jr,\alpha}-\overset{(+)}{h}_{,\alpha}u_j\left(x_1,x_2,\overset{(+)}{h}\right)+(-1)^r\overset{(-)}{h}_{,\alpha}u_j\left(x_1,x_2,\overset{(-)}{h}\right)\\
&-\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)}(a_{,\alpha}x_3-b_{,\alpha})\,P_r'(ax_3-b)\,u_j(x_1,x_2,x_3)dx_3\Bigg]\\
&+A_{ij3}\Bigg[u_j\left(x_1,x_2,\overset{(+)}{h}\right)-(-1)^ru_j\left(x_1,x_2,\overset{(-)}{h}\right)\\
&-a\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)}P_r'(ax_3-b)\,u_j(x_1,x_2,x_3)\,dx_3\Bigg]\\
&+B_{ij}u_{jr}+C_{ir}\,(x_1,x_2)=0,\quad (x_1,x_2)\in\omega,\quad i=\overline{1,n},\ r=0,1,\dots
\end{aligned}
$$

(under repeated $\alpha$ the sum from 1 to 2 is meant), i.e.,

$$
\begin{aligned}
&A_{ij\alpha}\left(u_{jr,\alpha}+\sum_{s=0}^{r}\overset{r}{a}_{\alpha s}u_{js}\right)+A_{ij3}\sum_{s=0}^{r}\overset{r}{a}_{3s}u_{js}+B_{ij}u_{jr}\\
&+A_{ij\alpha}\left[-\overset{(+)}{h}_{,\alpha}u_j\left(x_1,x_2,\overset{(+)}{h}\right)+(-1)^r\overset{(-)}{h}_{,\alpha}u_j\left(x_1,x_2,\overset{(-)}{h}\right)\right]\\
&+A_{ij3}\left[u_j\left(x_1,x_2,\overset{(+)}{h}\right)-(-1)^ru_j\left(x_1,x_2,\overset{(-)}{h}\right)\right]\\
&+C_{ir}\,(x_1,x_2)=0,\quad i=1,2,3,\quad r=0,1,\dots,
\end{aligned}
$$

where

$$
\overset{r}{a}_{\alpha r}:=r\frac{h_{,\alpha}}{h},\quad \overset{r}{a}_{\alpha s}:=(2s+1)\frac{\overset{(+)}{h}_{,\alpha}-(-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h},\quad s\neq r,\quad \alpha=1,2,
$$

$$
\overset{r}{a}_{3s}:=-(2s+1)\frac{1-(-1)^{s+r}}{2h}.
$$

The last system is the system of singular partial differential equations which can be easily rewritten in terms of $v_{ir}$. The obtained infinite system of partial differential equations will be a system with the order degeneration for a nonempty $\gamma_0$:

$$A_{ij\alpha}\left[(h^{r+1}v_{jr})_{,\alpha}+\sum_{s=0}^{r}\overset{r}{a}_{\alpha s}h^{s+1}v_{js}\right]+A_{ij3}\sum_{s=0}^{r}\overset{r}{a}_{3s}h^{s+1}v_{js}+B_{ij}h^{r+1}v_{jr}=F_{ir}, \qquad (3)$$

i.e.,

$$A_{ij\alpha}(h^{r+1}v_{jr})_{,\alpha}+\sum_{s=0}^{r}\overset{r}{E}_{ijs}h^{s+1}v_{js}=F_{ir}, \quad i=1,2,...,n, \ \ r=0,1,2,...,$$

where

$$\overset{r}{E}_{ijs}:=A_{ijk}\,\overset{r}{a}_{ks}+B_{ij}\delta_{rs},$$

$$\delta_{rs}=\begin{cases}1, & r=s;\\ 0, & r\neq s,\end{cases} \quad i,j=1,2,...n, \ \ s=0,1,...,r, \ \ r=0,1,2,...,$$

$$\begin{aligned}F_{ir}:=&A_{ij\alpha}\left[\overset{(+)}{h}_{,\alpha}u_j\left(x_1,x_2,\overset{(+)}{h}\right)-(-1)^r\overset{(-)}{h}_{,\alpha}u_j\left(x_1,x_2,\overset{(-)}{h}\right)\right]\\ &-A_{ij3}\left[u_j\left(x_1,x_2,\overset{(+)}{h}\right)-(-1)^r u_j\left(x_1,x_2,\overset{(-)}{h}\right)\right]-C_{ir}\left(x_1,x_2\right),\\ &\qquad i=1,2,...,n, \ \ r=0,1,2,\cdots.\end{aligned} \qquad (4)$$

Those of $u_j(x_1,x_2,\overset{(+)}{h})$, $u_j(x_1,x_2,\overset{(-)}{h})$ which are given in $3D$ problem on $x_3=\overset{(+)}{h}(x_1,x_2)$ and $x_3=\overset{(-)}{h}(x_1,x_2)$ remain with its given boundary values in the right hand side $F_{ir}$ of the system (3), those of $u_j(x_1,x_2,\overset{(+)}{h})$, $u_j(x_1,x_2,\overset{(-)}{h})$ which are not given on the above surfaces should be replaced by their Legendre-Fourier expansions there, i.e.,

$$u_j(x_1,x_2,\overset{(\pm)}{h})=\sum_{s=0}^{\infty}(\pm 1)^s(s+\frac{1}{2})h^s v_{js}(x_1,x_2),$$

containing unknown functions $v_{js}(x_1,x_2)$. The last terms are to be transferred to the left hand side of the system (3), since they contain unknown functions which are sought for.

On the lateral subsurface

$$\Gamma:=\{(x_1,x_2,x_3)\in R^3 \ : \ (x_1,x_2)\in\partial\omega, \ \ \overset{(-)}{h}(x_1,x_2)\leq x_3\leq\overset{(+)}{h}(x_1,x_2)\}$$

of $\partial\Omega$ the boundary conditions should be reformulated as follows:

($i$) where $\overset{(+)}{h}(x_1,x_2)>\overset{(-)}{h}(x_1,x_2)$, the functions $v_{jr}$ should be calculated by given $u_j(x_1,x_2,x_3)|_{(x_1,x_2)\in\partial\omega}$ by means of the formulas

$$v_{jr}(x_1,x_2)=\frac{1}{h^{r+1}(x_1,x_2)}\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)}u_j(x_1,x_2,x_3)\,P_r(ax_3-b)\,dx_3, \quad (x_1,x_2)\in\partial\omega; \qquad (5)$$

(*ii*) where $\overset{(+)}{h}(x_1, x_2) = \overset{(-)}{h}(x_1, x_2)$, i.e., on the cusped (in particular, cuspidal) edge, depending on the sharpening geometry of the cusped edge, the unknown functions $v_{jr}$ either should be prescribed or not, but how to calculate them from boundary conditions of $3D$ problem is the subject of special investigation.

The system (1), in particular, contains the governing first order system of the linear theory of elasticity with respect to the stress tensor and displacement vector components. This approach is already successfully applied to the investigation of cusped prismatic shells with cuspidal edges (see [3-7]).

This method can be also applied to the systems of higher order partial differential equations as a method of dimension reduction from $R^m$ to $R^{m-1}$, $m \geq 2$.

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