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UNIQUENESS AND EXISTENCE THEOREMS OF STATICS BVPs OF THE THEORY OF CONSOLIDATION WITH DOUBLE POROSITY

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Abstract. The purpose of this paper is to consider two-dimensional version of statics of the Aifantis' equation of the theory of consolidation with double porosity and to study the uniqueness and existence of solutions of basic boundary value problems (BVPs).

In this work we intend to extend potential method and the theory of integral equation to BVPs of the theory of consolidation with double porosity. The potential method and the theory of integral equation are applied to the investigation of the first and second BVPs of statics of the theory of consolidation with double porosity. For their problems we construct Fredholm type integral equations. Using these equations, the potential method and generalized Green's Formulas, we prove the existence and uniqueness theorems of solutions for the first and second BVPs for the bounded and unbounded domains. For the Aifantis' equation of statics we construct one particular solution and we reduce the solution of basic BVPs of the theory of consolidation with double porosity to the solution of the basic BVPs for the equation of an isotropic body.

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AMS subject classification (2000): 74G25; 74G30.

1. Introduction

In a material with two degrees of porosity, there are two pore system, the primary and the secondary. For example in a fissured rock (i.e., a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or secondary porosity. When fluid flow and deformations processes occur simultaneously, three coupled partial differential equations can be derived [1, 2] to describe the relationships governing pressure in the primary and secondary pores (and therefore the mass exchange between them) and the displacement of the solid.

A theory of consolidation with double porosity has been proposed by Aifantis. The physical and mathematical foundations of the theory of double porosity were considered in the papers [1-3], where analytical solutions of the relevant equations are also given. In part I of a series of paper on the subject, Wilson and Aifantis [1] gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory. They also solved several representative boundary value problems. In part II of this series, uniqueness and variational principles were established by Beskos and Aifantis [2] for the equations of double porosity, while in part III Khaled, Beskos and Aifantis [3] provided a related finite element to consider the numerical solution of Aifantis' equations of double porosity (see [1-3] and references cited therein).The basic results and the historical information on the theory of porous media were summarized by de Boer [4].

In this work we prove the existence and uniqueness theorems of solutions of basic BVPs of the theory of consolidation with double porosity for bounded and unbounded domains. We used the potential method for the proof of all theorems. The basic results on this method are given in [6].

2. Basic equations and boundary value problems

Let $\mathbf{x} = (x_1, x_2)$ be the point of the Euclidean two-dimensional space E^2. The basic equations of statics of the theory of consolidation with double porosity in the case of plane deformation have the following form [1-2]

$$\begin{aligned} &\mathbf{B}(\partial x)\mathbf{u} = \mu\,\Delta\mathbf{u} + (\lambda+\mu) grad\, div\,\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) = 0, \\ &(m_1\Delta - k)p_1 + kp_2 = 0, \qquad kp_1 + (m_2\Delta - k)p_2 = 0, \end{aligned} \tag{1}$$

where $\mathbf{u} = (u_1, u_2)$ is the displacement vector, p_1 is the fluid pressure within the primary pores and p_2 is the fluid pressure within the secondary pores, $m_j = \frac{k_j}{\mu^*}, j = 1, 2$. The constant λ is the Lame modulus, μ is the shear modulus, the constants β_1 and β_2 are measure the change of porosities due to an applied volumetric strain. The constant μ^* denotes the viscosity of the pore fluid, the constant k measures the transfer of fluid from the secondary pores to the primary pores. The quantities λ, μ, β_j, k $(j = 1, 2)$ and μ^* are all positive constants. $\triangle = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$ is two-dimensional Laplace operator.

The equation (1) can be written in matrix-vector form

$$\mathbf{A}(\partial x)\mathbf{U}(x) = 0, \tag{2}$$

where $\mathbf{U}(x) = (u_1, u_2, p_1, p_2)$,

$$\begin{aligned} &\mathbf{A}(\partial x) = \parallel A_{pq}(\partial x) \parallel_{4x4},\ A_{jj}(\partial x) = \mu\Delta + (\lambda+\mu)\frac{\partial^2}{\partial x_j^2}, \\ &A_{12}(\partial x) = A_{21}(\partial x) = (\lambda+\mu)\frac{\partial^2}{\partial x_1 \partial x_2}, \\ &A_{j3}(\partial x) = -\beta_1\frac{\partial}{\partial x_j}, \quad A_{j4}(\partial x) = -\beta_2\frac{\partial}{\partial x_j}, \\ &A_{3j}(\partial x) = 0,\ A_{4j}(\partial x) = 0, \qquad A_{33}(\partial x) = m_1\Delta - k, \\ &A_{34}(\partial x) = A_{43}(\partial x) = k, \qquad A_{44}(\partial x) = m_2\Delta - k, \qquad j = 1, 2. \end{aligned}$$

Let $D^+(D^-)$ be a bounded (an unbounded) two-dimensional domain surrounded by the contour S. $\overline{D^+} = D^+ \cup S$, $D^- = E_2 \setminus \overline{D^+}$. Suppose that $S \in C^{1,\alpha}$, $0 < \alpha \le 1$.

First of all we introduce the definition of a regular vector-function.

Definition 1. A vector-function $\mathbf{U} = (u_1, u_2, p_1, p_2)$ defined in D^+ (or in D^-) is called regular if $U \in C^2(D^+)\bigcap C^1(\overline{D^+})$ (or $\mathbf{U} \in C^2(D^-)\bigcap C^1(\overline{D^-})$) and in the unbounded domain D^- the vector U additionally satisfies the following conditions at infinity:

$$\mathbf{U}(x) = o(1), \quad \frac{\partial \mathbf{U}_k}{\partial x_j} = O(|x|^{-2}), \quad |x|^2 = x_1^2 + x_2^2, \;\; j = 1, 2. \tag{3}$$

The internal and external basic BVPs are formulated as follows:

Find a regular vector $\mathbf{U}$ satisfying in $D^+(D^-)$ the equation (1) and on the boundary S one of the following conditions is given:

Problem $(I)_f^{\pm}$. The displacement vector and the fluid pressures are given on S :

$$\mathbf{u}^{\pm} = \mathbf{f}^{\pm}(z), \qquad p_1^{\pm} = f_3^{\pm}(z), \qquad p_2^{\pm} = f_4^{\pm}(z), \qquad z \in S,$$

Problem $(II)_f^{\pm}$. The stress vector and the normal derivatives of the preasure functions $\dfrac{\partial p_j}{\partial n}$, $\quad j = 1, 2$, are given on S :

$$[\mathbf{P}(\partial x, n)\mathbf{u}]^{\pm} = \mathbf{f}^{\pm}(z), \quad \left(\frac{\partial p_1}{\partial n}\right)^{\pm} = f_3^{\pm}(z), \qquad \left(\frac{\partial p_2}{\partial n}\right)^{\pm} = f_4^{\pm}(z), \qquad z \in S.$$

Problem $(III)_f^{\pm}$. The displacement vector and the normal derivatives of the pressure functions $\dfrac{\partial p_j}{\partial n}$, $\quad j = 1, 2$, are given on S :

$$\mathbf{u}^{\pm} = \mathbf{f}^{\pm}(z), \qquad \left(\frac{\partial p_1}{\partial n}\right)^{\pm} = f_3^{\pm}(z), \qquad \left(\frac{\partial p_2}{\partial n}\right)^{\pm} = f_4^{\pm}(z), \qquad z \in S.$$

Problem $(IV)_f^{\pm}$. The stress vector and the fluid pressures are given on S :

$$[\mathbf{P}(\partial x, n)\mathbf{u}]^{\pm} = \mathbf{f}^{\pm}(z), \qquad p_1^{\pm} = f_3^{\pm}(z), \qquad p_2^{\pm} = f_4^{\pm}(z), \qquad z \in S,$$

where $(.)^+$ denotes the limiting value from D^+, $(.)^-$ denotes the limiting value from D^- and $\mathbf{f} = (f_1, f_2), f_3, f_4$ are the given functions, $\mathbf{P}(\partial x, n)\mathbf{u}$ is a stress vector which acts on the elements of the arc with the exterior to D^+ unit normal vector $\mathbf{n} = (n_1, n_2)$ at the point $x \in S$,

$$\mathbf{P}(\partial x, n)\mathbf{u} = \mathbf{T}(\partial x, n)\mathbf{u} - \mathbf{n}(\beta_1 p_1 + \beta_2 p_2), \tag{4}$$

and [6]

$$\begin{aligned} &\mathbf{T}(\partial x, n) = \| \, T_{kj} \, \|_{2x2}, \\ &T_{kj}(\partial x, n) = \mu\delta_{kj}\frac{\partial}{\partial n} + \lambda n_k \frac{\partial}{\partial x_j} + \mu n_j \frac{\partial}{\partial x_k}, \\ &\frac{\partial}{\partial n} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2}, \qquad k, j, = 1, 2. \end{aligned} \tag{5}$$

Now we introduce the generalized stress vector $\overset{\kappa}{\mathbf{P}}(\partial x, n)\mathbf{u}$, where

$$\overset{\kappa}{\mathbf{P}}(\partial x, n)\mathbf{u} = \overset{\kappa}{\mathbf{T}}(\partial x, n)\mathbf{u} - \mathbf{n}(\beta_1 p_1 + \beta_2 p_2),$$

κ is an arbitrary positive constant and

$$\overset{\kappa}{\mathbf{T}}(\partial x, n)\mathbf{u} = (2\mu - \kappa)\frac{\partial \mathbf{u}}{\partial n} + (\lambda + \kappa)\mathbf{n} div \mathbf{u} + (\kappa - \mu)\mathbf{s}\omega,$$
$$\mathbf{s} = \begin{pmatrix} -n_2 \\ n_1 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \qquad omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}. \tag{6}$$

If $\kappa = 0$ from (6) we have $\overset{\kappa}{\mathbf{T}}(\partial x, n)\mathbf{u} = \mathbf{T}(\partial x, n)\mathbf{u}$. We set $\overset{\kappa}{\mathbf{T}}(\partial x, n)\mathbf{u} = \mathbf{N}(\partial x, n)\mathbf{u}$ for $\kappa = \frac{\mu(\lambda+\mu)}{\lambda+3\mu}$.

3. Generalized Green's formulas

Let us write the generalized Green's formulas for the domains D^+ and D^-. Let $\mathbf{u}(x)$ be a regular solution of equation (1) in D^+. Multiply the first equation of (1) by $\mathbf{u}(x)$. Integration the result over D^+ and apply the integration by parts formula to obtain

$$\int_{D^+} \mathbf{u}\mathbf{B}(\partial x)\mathbf{u} d\sigma = \int_S \mathbf{u}\overset{\kappa}{\mathbf{P}}(\partial x, n)\mathbf{u} ds - \int_{D^+} \left[\overset{\kappa}{\mathrm{E}}(\mathbf{u},\mathbf{u}) - (\beta_1 p_1 + \beta_2 p_2) div \mathbf{u}\right] d\sigma.$$

If the vector $\mathbf{u}$ is a solution of homogeneous equation $\mathbf{B}(\partial x)\mathbf{u} = 0$, then the last equation gives

$$\int_{D^+} \left[\overset{\kappa}{\mathrm{E}}(\mathbf{u},\mathbf{u}) - (\beta_1 p_1 + \beta_2 p_2) div \mathbf{u}\right] d\sigma = \int_S \mathbf{u}\overset{\kappa}{\mathbf{P}}(\partial x, n)\mathbf{u}\ ds, \tag{7}$$

where

$$2\overset{\kappa}{\mathrm{E}}(\mathbf{u},\mathbf{u}) = (2\lambda + 2\mu - \kappa)(div \mathbf{u})^2 + (2\mu - \kappa)\left[\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2\right] + \frac{\kappa}{2}\omega^2.$$

For the positive definiteness of the potential energy the inequality $0 < \kappa \leq 2\mu$ is necessary and sufficient. Obviously the potential energy $E(\mathbf{u},\mathbf{u})$ is obtained from $\overset{\kappa}{\mathrm{E}}(\mathbf{u},\mathbf{u})$ if we set $\kappa = 0$.

If the vector $\mathbf{u}(x)$ satisfies the conditions (3) the Green's formula for the region D^- takes the form

$$\int_{D^-} \left[\overset{\kappa}{\mathrm{E}}(\mathbf{u},\mathbf{u}) - (\beta_1 p_1 + \beta_2 p_2) div\ \mathbf{u}\right] d\sigma = -\int_S \mathbf{u}\overset{\kappa}{\mathbf{P}}(\partial x, n)\mathbf{u} ds. \tag{8}$$

Analogously we obtain the Green's formula for p_j, $j = 1, 2$,

$$\int_{D^+} \left[m_1(grad p_1)^2 + m_2(grad p_2)^2 + k(p_1 - p_2)^2\right] d\sigma = \int_S \left[m_1 p_1 \frac{\partial p_1}{\partial n} + m_2 p_2 \frac{\partial p_2}{\partial n}\right] ds, \tag{9}$$

$$\int_{D^+}\left[m_1(gradp_1)^2+m_2(gradp_2)^2+k(p_1-p_2)^2\right]d\sigma=-\int_S\left[m_1p_1\frac{\partial p_1}{\partial n}+m_2p_2\frac{\partial p_2}{\partial n}\right]ds. \tag{10}$$

Remark. Note that if $\beta_1p_1+\beta_2p_2=const$, in view of the equality $\int\limits_{D^+}div\mathbf{u}=\int\limits_S\mathbf{nu}ds$, from (7) we get

$$\int_{D^+}\overset{\kappa}{\mathrm{E}}(\mathbf{u,u})d\sigma=\int_S\mathbf{u}\overset{\kappa}{\mathrm{T}}(\partial x,n)\mathbf{u}\;ds. \tag{11}$$

4. The uniqueness theorems

In this subsection we prove the uniqueness theorems of solutions to the above formulated problems. Let above formulated problems have two regular solutions $\mathbf{U}^{(1)}(x)$ and $\mathbf{U}^{(2)}(x)$, where $\mathbf{U}^{(k)}(x)=(u_1^{(k)},u_2^{(k)},p_1^{(k)},p_2^{(k)}),\quad k=1,2.$ Let's consider

$$\mathbf{U}(x)=\mathbf{U}^{(1)}(x)-\mathbf{U}^{(2)}(x).$$

Evidently, the vector $\mathbf{U}(x)$ satisfies (1) and the homogeneous boundary conditions $(\mathbf{f}=0,\quad f_3=0,\quad f_4=0)$.

Now we prove the following theorems:

Theorem 1. *The first internal boundary value problem $(I)_f^+$ admit at most one regular solution in the domain D^+.*

Proof. Evidently, the vector $\mathbf{U}(x)$ satisfies the system (1) and the boundary condition $\mathbf{U}(x)=0$ on S. From (9) we obtain $p_1=p_2=c,\quad x\in D^+$. Since $p_k^+=0$, we have $c=0$, and $p_1=p_2=0,\quad x\in D^+$. Note that if $\mathbf{u}$ is a regular solution of the equation (1), we have Green's formula (7). Using (7), when $\kappa=0$ and taking into account the fact that the potential energy is positive definite, we conclude that $u_1=c_1-\epsilon x_2,\quad u_2=c_2+\epsilon x_1\quad x\in D^+$, where $\epsilon,\ c=const$. Since $\mathbf{U}^+=0$, we have $c=0,\quad \epsilon=0$ and $\mathbf{u}(x)=0,\quad x\in D^+$.

Theorem 2. *The first external boundary value problem $(I)_f^-$ has at most one regular solution in the domain D^-.*

Proof. The vectors $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$ in the domain D^- must satisfy the condition (3). In this case the formulas (8)-(10) are valid and $\mathbf{U}(x)=\mathbf{C},\quad x\in D^-$, where $\mathbf{C}$ is again the constant vector. But $\mathbf{U}$ on the boundary satisfies the condition $\mathbf{U}^-=0$, which implies that $\mathbf{C}=0$ and $\mathbf{U}(x)=0,\quad x\in D^-$.

Analogously the following theorems can be proved :

Theorem 3. *If the condition $0<\kappa\le 2\mu$ is satisfied then any two regular solutions of the second internal boundary value problem $(II)_f^+$ may differ only to within additive vector $\boldsymbol{V}=(\boldsymbol{u},p_1,p_2)$, where*

$$u_1=c_1-\epsilon x_2+c_1x_1,\quad u_2=c_2+\epsilon x_1+c_1x_2\quad p_k=c,\quad x\in D^+,$$

ϵ and c are arbitrary real constants and $c_1 = \frac{c(\beta_1+\beta_2)}{2(\lambda+\mu)}$.

Theorem 4. *The boundary value problems $(II)_f^-$, $(III)_f^-$, $(IV)_f^-$ admit at most one regular solution in the domain D^-.*

Theorem 5. *Two regular solutions of the $(III)_f^+$ boundary value problem in the domain D^+ may differ by the vector $\boldsymbol{V} = (\boldsymbol{u}, p_1, p_2)$, where $\boldsymbol{u} = 0$, and $p_1 = p_2 = c$.*

Theorem 6. *Two regular solutions of the $(IY)_f^+$ boundary value problem may differ by the vector $\boldsymbol{V}(\boldsymbol{u}, p_1, p_2)$, where $\boldsymbol{u}$ vector is a rigid displacement and $p_1 = p_2 = 0$.*

5. An existence theorems

In this section we establish the existence of regular solutions of the basic BVPs $(I)_f^\pm$ and $(II)_f^\pm$ by means of the potential method and the theory of singular integral equations.

Problem $(I)_f^+$. First let us show that the nonhomogeneous system

$$(m_1\Delta - k)p_1 + kp_2 = F_3(x), \quad kp_1 + (m_2\Delta - k)p_2 = F_4(x) \tag{12}$$

always reduces to the homogeneous system by seeking one particular solution. We choose more simple method for constructing particular solution. A solution p_k, $k = 1, 2$ is sought in the form

$$p_1 = -\frac{m_2}{2\pi s^2}\int\limits_{D^+} [K_0(sr) + \ln r]\, F_3(y)d\sigma, \quad p_2 = \frac{m_1}{2\pi s^2}\int\limits_{D^+} [K_0(sr) + \ln r]\, F_4(y)d\sigma, \tag{13}$$

where [5]

$$K_0(sr) = -I_0(sr)\left(\ln\frac{sr}{2} + C\right) - 2\sum_{k=1}^{\infty}\frac{1}{(k!)^2}\left(\frac{sr}{2}\right)^{2k}\left(\frac{1}{k} + \frac{1}{k-1} + \dots + 1\right),$$
$$I_0(sr) = \sum_{k=1}^{\infty}\frac{1}{(k!)^2}\left(\frac{sr}{2}\right)^{2k}, \quad s^2 = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right), \; r^2 = (x_1 - y_1)^2 + (x_1 - y_1)^2.$$

It is obvious that integrand in (13) contains the terms $r^{2k}\ln r$, $k = 1, 2, \dots$ and we can write

$$(\Delta - s^2)p_1 = \frac{m_2}{2\pi}\int\limits_{D^+}\ln r F_3(y)d\sigma, \quad (\Delta - s^2)p_2 = -\frac{m_1}{2\pi}\int\limits_{D^+}\ln r F_4(y)d\sigma.$$

From this we get

$$\Delta(\Delta - s^2)p_1 = m_2F_3(x), \quad \Delta(\Delta - s^2)p_2 = -m_1F_4(x).$$

Thus we obtain that the particular solutions of the equation (12) are

$$\begin{aligned} p_1 &= \frac{1}{2\pi m_2 s^2}\int\limits_{D^+}[K_0(sr) + \ln r]F_3(y)d\sigma, \\ p_2 &= -\frac{1}{2\pi m_1 s^2}\int\limits_{D^+}[K_0(sr) + \ln r]F_4(y)d\sigma. \end{aligned} \tag{14}$$

At first let's search the fundamental solution of the following equation

$$(m_1\Delta - k)p_1 + kp_2 = 0, \quad kp_1 + (m_2\Delta - k)p_2 = 0. \tag{15}$$

It is obvious that

$$\begin{pmatrix} m_1\Delta - k & k \\ k & m_2\Delta - k \end{pmatrix}\begin{pmatrix} m_2\Delta - k & -k \\ -k & m_1\Delta - k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} m_1m_2\Delta(\Delta - s^2), \tag{16}$$

where $s^2 = k\left(\dfrac{1}{m_1} + \dfrac{1}{m_2}\right)$. The fundamental solution of the equation $\Delta(\Delta - s^2)\psi = 0$ is

$$\psi = \alpha_0 K_0(sr) + \alpha_1 \ln r.$$

For the unknown coefficient α_j we obtain the following equations

$$-\alpha_0 + \alpha_1 = 0, \ \ \alpha_0 s^2 = 1.$$

from here we obtain $\alpha_0 = \alpha_1 = \dfrac{1}{s^2}$ and $\psi = \dfrac{1}{s^2}(K_0(sr) + \ln r)$. Obviously $\Delta\psi$ contains a logarithmic singularity as $x \to y$.

From the reduced discussion it is evident that the fundamental matrix of the equation (15) must have the form

$$\mathbf{\Gamma^{(1)}}(x-y) = \begin{pmatrix} m_2K_0(sr) - \dfrac{k}{s^2}[K_0(sr) + \ln r] & -\dfrac{k}{s^2}[K_0(sr) + \ln r] \\ -\dfrac{k}{s^2}[K_0(sr) + \ln r] & m_1K_0(sr) - \dfrac{k}{s^2}[K_0(sr) + \ln r] \end{pmatrix}. \tag{17}$$

The matrix $\mathbf{\Gamma^{(1)}}(x-y)$ has a logarithmic singularity as $x \to y$. It is evident that every column of the matrix $\mathbf{\Gamma^{(1)}}(x-y)$ is a solution of the system (15) with respect to the point x , if $x \neq y$.

First let us prove the existence of solution of the first BVP ($p_1^+ = f_4^+, \quad p_2^+ = f_5^+,$) for the equation (15) in the domain D^+. A solution will be sought in the form of the double layer potential

$$\mathbf{p}(x) = \begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} = \frac{1}{\pi}\int_S \frac{\partial}{\partial n}\mathbf{\Gamma^{(1)}}(y-x)\mathbf{g}(y)ds_y, \quad x \in D^+. \tag{18}$$

Passing the limit as $x \to z \in S$ and taking into account the boundary condition, for determining the unknown vector function $\mathbf{g}(y) = (g_3, g_4)$, we obtain the following Fredholm integral equation of the second kind

$$-m_2g_3(z) + p_1(z) = f_3^+(z), \quad -m_2g_4(z) + p_2(z) = f_4^+(z), \tag{19}$$

where $f_j^+(z), \quad j = 3, 4$, are given continuous functions and

$$\begin{aligned} p_1(z) &= \frac{1}{\pi}\int_S \frac{\partial}{\partial n}\left[m_2K_0(sr)g_3(y) - \frac{k}{s^2}(K_0(sr) + \ln r)(g_3(y) + g_4(y))\right]ds_y, \\ p_2(z) &= \frac{1}{\pi}\int_S \frac{\partial}{\partial n}\left[m_2K_0(sr)g_4(y) - \frac{k}{s^2}(K_0(sr) + \ln r)(g_3(y) + g_4(y))\right]ds_y. \end{aligned} \tag{20}$$

Let us prove that the equation (19) is solvable for any continuous right-hand side. Let us prove that the homogeneous version of (19) ($f_j = 0$) has only the trivial solution. Let the vector $\mathbf{g} \neq 0$ be some solution to it. Obviously $(p_j)^+ = 0, \quad j = 1, 2$. Using Green's formula in D^+

$$\int\limits_{D^+} \left[m_1(gradp_1)^2 + m_2(gradp_2)^2 + k(p_1 - p_2)^2\right] ds = \int\limits_S \left[m_1 p_1 \frac{\partial p_1}{\partial n} + m_2 p_2 \frac{\partial p_2}{\partial n}\right] ds,$$

we obtain $p_1 = p_2 = c, \ x \in D^+$. ($c$ is an arbitrary constant). It is easy to show that g_j has a continuous derivative, then we have the following formula

$$0 = \left(\frac{\partial p_1}{\partial n}\right)^+ = \left(\frac{\partial p_1}{\partial n}\right)^-, \quad 0 = \left(\frac{\partial p_2}{\partial n}\right)^+ = \left(\frac{\partial p_2}{\partial n}\right)^-.$$

Using Green's formula in D^-, we obtain $p_1 = p_2 = c_1$, where c_1 is an arbitrary constant, i.e. we have $(p_1)^+ - (p_1)^- = -2m_2 g_3, \quad (p_2)^+ - (p_2)^- = -2m_1 g_4$. If we substitute the last identity in (20), after elementary transformation we obtain $g_3 = \dfrac{c - c_1}{2m_2}, \quad g_4 = \dfrac{c - c_1}{2m_1}$ and (18) takes the form

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{1}{\pi} \int\limits_S \frac{\partial lnr}{\partial n} \begin{pmatrix} -1 \\ -1 \end{pmatrix} (c - c_1) ds = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} (c - c_1).$$

From here we get $c = c_1, \quad g_3 = g_4 = 0$ and hence the homogeneous equation $(19)_0$ corresponding to the equation (19) has only the trivial solution. This implies that the equation (19) is solvable for any continuous right-hand side.

Remark. Analogously we prove the existence of solution of external first BVP ($p_1^- = f_3^-, \quad p_2^- = f_4^-,$) for the equation (15) in the domain D^-. A solution of the first boundary value problem has the form

$$\mathbf{P}(x) = \frac{1}{\pi} \int\limits_S \frac{\partial}{\partial n} \mathbf{\Gamma}^{(1)}(y - x) \mathbf{g}(y) ds_y, \quad x \in D^-, \tag{21}$$

where $g(y)$ is a solution of the following Fredholm integral equation of the second kind

$$m_2 g_1(z) + p_1(z) = f_3^-(z), \quad m_2 g_2(z) + p_2(z) = f_4^-(z), \tag{22}$$

$f_j(z), \ \ j = 3, 4$, are given continuous functions and $p_j, \ \ j = 1, 2$, are given by (20).

Further we assume that $\mathbf{P}(x)$ is known, when $x \in D^+$ or $x \in D^+$(see (18) and (21)). Substitute the $\beta_1 p_1 + \beta_2 p_2$ in (1). Let's search the particular solution of the following equation

$$\mu \Delta \mathbf{u} + (\lambda + \mu) graddiv \mathbf{u} = grad(\beta_1 p_1 + \beta_2 p_2). \tag{23}$$

We put

$$\mathbf{u}_0 = \frac{1}{\pi} \int\limits_D \mathbf{\Gamma}(x - y) grad(\beta_1 p_1 + \beta_2 p_2) ds, \tag{24}$$

where [6]

$$\mathbf{\Gamma}(x-y)=\begin{pmatrix} \dfrac{\lambda+3\mu}{2a\mu}\ln r-\dfrac{\lambda+\mu}{2a\mu}\left(\dfrac{\partial r}{\partial x_1}\right)^2, & -\dfrac{\lambda+\mu}{2a\mu}\dfrac{\partial r}{\partial x_1}\dfrac{\partial r}{\partial x_2} \\ -\dfrac{\lambda+\mu}{2a\mu}\dfrac{\partial r}{\partial x_1}\dfrac{\partial r}{\partial x_2}, & \dfrac{\lambda+3\mu}{2a\mu}\ln r-\dfrac{\lambda+\mu}{2a\mu}\left(\dfrac{\partial r}{\partial x_2}\right)^2 \end{pmatrix}.$$

Substituting the volume potential $\mathbf{u}_0$ into (23), we obtain [6]

$$\mu\Delta\mathbf{u}_0+(\lambda+\mu)graddiv\mathbf{u}_0=grad(\beta_1p_1+\beta_2p_2). \tag{25}$$

Thus we have proved that $\mathbf{u}_0$ is a particular solution of the equation (23). In (24) D denotes either D^+ or D^-, $grad(\beta_1p_1+\beta_2p_2)$ is a continuous vector in D^+ along with its first order derivatives. When $D=D^-$, the vector $grad(\beta_1p_1+\beta_2p_2)$ has to satisfy the following decay condition at infinity

$$grad(\beta_1p_1+\beta_2p_2)=O(|x|^{-2-\alpha}), \quad \alpha>0.$$

Thus the general solution of the equation (23) is representable in the form $\mathbf{u}=\mathbf{V}+\mathbf{u}_0$, where

$$\mu\Delta\mathbf{V}+(\lambda+\mu)graddiv\mathbf{V}=0. \tag{26}$$

This equation is the equation of an isotropic elastic body. Thus we have reduced the solution of basic BVPs of the theory of consolidation with double porosity to the solution of the basic BVPs for the equation of an isotropic elastic body.

The solution of the first BVP ($V^+=F^+$) is given in the form [6]

$$\mathbf{V}(x)=\frac{1}{\pi}\int\limits_S \mathbf{N}(\partial y,n)\mathbf{\Gamma}(x-y)\mathbf{g}(y)ds, \tag{27}$$

where

$$\mathbf{N}(\partial y,n)\mathbf{\Gamma}(x-y)=\begin{pmatrix} 1+\dfrac{\lambda+\mu}{\lambda+3\mu}\cos 2\theta, & \dfrac{\lambda+\mu}{\lambda+3\mu}\sin 2\theta \\ \dfrac{\lambda+\mu}{\lambda+3\mu}\sin 2\theta, & 1-\dfrac{\lambda+\mu}{\lambda+3\mu}\cos 2\theta \end{pmatrix}\frac{\partial\theta}{\partial s},$$

$$\theta=\arctan\frac{y_2-x_2}{y_1-x_1}, \quad \frac{\partial}{\partial s}=n_1\frac{\partial}{\partial x_2}-n_2\frac{\partial}{\partial x_1},$$

$\mathbf{g}$ is a solution of Fredholm integral equation of the second kind

$$\mathbf{g}(z)+\frac{1}{\pi}\int\limits_S \mathbf{N}(\partial y,\mathbf{n})\mathbf{\Gamma}(y-z)\mathbf{g}(y)ds=\mathbf{f}^+(z). \tag{28}$$

To prove the regularity of the double layer potential in the domain D^+, it is sufficient to assume that $S\in C^{2,\beta}$, $0<\beta<1$, $f\in C^{1,\alpha}(S)$, $0<\alpha<\beta$.

We have thereby proved the following theorem.

Theorem 7. *If* $S \in C^{2,\beta}$, $\quad 0 < \beta < 1$, $\quad f_3, f_4, \boldsymbol{f} \in C^{1,\alpha}(S)$, $\quad 0 < \alpha < \beta$, *then a regular solution of problem* $(I)_f^+$ *exists, it is unique and represented by the potential of double-layer (18) and (27), where* $\boldsymbol{g}$ *is a solution of the Fredholm integral equations (19) and (28) respectively which are always solvable for arbitrary functions* f_3, f_4, *and* $\boldsymbol{f}$*.*

Problem $(I)_f^-$**.** Now consider the first BVP $(\mathbf{V}^-(z) = \mathbf{f}^-(z))$ in the domain D^-. The solution is sought in the form [6]

$$\mathbf{V}(x) = \frac{1}{\pi} \int\limits_S \left[\mathbf{N}(\partial y, n)\mathbf{\Gamma}(x-y) + \frac{1}{2}\mathbf{N}(\partial y, n)\mathbf{\Gamma}(y) \right] \mathbf{g}(y) ds. \tag{29}$$

For determining the unknown vector $\mathbf{g}$ we obtain the following Fredholm integral equation of the second kind

$$-\mathbf{g}(z) + \frac{1}{\pi} \int\limits_S \left[\mathbf{N}(\partial y, \mathbf{n})\mathbf{\Gamma}(y-z) + \frac{1}{2}\mathbf{N}(\partial y, n)\mathbf{\Gamma}(y) \right] \mathbf{g}(y) ds = \mathbf{f}^-(z). \tag{30}$$

Here we assume that $\int\limits_S \mathbf{g}(y) ds = 0$ which implies the single layer potential vanishing at infinity.

The equation (30) is always solvable if the condition $\int\limits_S \mathbf{g}(y) ds = \int\limits_S \mathbf{f}(y) ds = 0$ is fulfilled [6].

To prove the regularity of the potential defined by (29) in the domain D^-, it is sufficient to assume that $S \in C^{2,\beta}$, $\quad 0 < \beta < 1$, $\quad \mathbf{f} \in C^{1,\alpha}(S)$, $\quad 0 < \alpha < \beta$.

Theorem 8. $S \in C^{2,\beta}$, $\quad 0 < \beta < 1$, $\quad f_3, f_4, \boldsymbol{f} \in C^{1,\alpha}(S)$, $\quad 0 < \alpha < \beta$, *then a regular solution of problem* $(I)_f^-$ *exists, it is unique and represented by the potentials of double-layer (21) and (29), where* $\boldsymbol{g}$ *is a solution of the Fredholm integral equations (22) and (30) respectively which are always solvable for an arbitrary right hand side.*

Thus we have proved the solvability of the first boundary value problem in the domains D^+ and D^-.

Problem $(II)_f^+$**.** A solution of BVP $\left(\dfrac{\partial p_1}{\partial n}\right)^+ = f_3(z)$, $\quad \left(\dfrac{\partial p_2}{\partial n}\right)^+ = f_4(z)$ of the equation (15) will be sought in the form

$$\mathbf{p}(x) = \frac{1}{\pi} \int\limits_S \mathbf{\Gamma}^{(1)}(x-y) \begin{pmatrix} g_3(y) \\ g_4(y) \end{pmatrix} ds_y, \tag{31}$$

where $\mathbf{\Gamma}^{(1)}(x-y)$ is given by formula (17), $S \in C^{1,\beta}, 0 < \beta \le 1$ is a closed Lyapunow curve, $g_k, k = 3, 4$, are unknown functions.

Taking into account the boundary conditions for determining the functions g_k,we obtain Fredholm integral equations of the second kind

$$m_2 g_3(z) + \frac{\partial p_1(z)}{\partial n} = f_3(z), \quad m_1 g_4(z) + \frac{\partial p_2(z)}{\partial n} = f_4(z), \quad z \in S. \tag{32}$$

The origin is assumed to be in the domain D^+. Let us prove that the equation (32) is always solvable. To this end, we consider the homogeneous equation obtained from (32) for $f_j = 0$ and prove that it has only the trivial solution. Let $g_0 \neq 0$ be any solution of this equation. Since $f_j = 0$, we have

$$\left(\frac{\partial p_1}{\partial n}\right)^+ = 0, \quad \left(\frac{\partial p_2}{\partial n}\right)^+ = 0.$$

Using Green's formula (9), we obtain

$$p_k = c = const, \quad k = 1, 2, \quad x \in D^+. \tag{33}$$

But the potential (31) is a continuous function when the point x tends to any point z of the boundary and we get $p_k(x) = c, \quad x \in D^-$. From last conditions it follows that

$$0 = \left(\frac{\partial p_1}{\partial n}\right)^+ = \left(\frac{\partial p_1}{\partial n}\right)^-, \quad 0 = \left(\frac{\partial p_2}{\partial n}\right)^+ = \left(\frac{\partial p_2}{\partial n}\right)^-,$$

$$0 = \left(\frac{\partial p_1}{\partial n}\right)^+ - \left(\frac{\partial p_1}{\partial n}\right)^- = 2m_2 g_3, \quad 0 = \left(\frac{\partial p_2}{\partial n}\right)^+ - \left(\frac{\partial p_2}{\partial n}\right)^- = 2m_1 g_4.$$

Finally we conclude that the homogeneous equation, corresponding to the equation (32) has only the trivial solution. Thus the equation (32) is always solvable for any continuous right-hand side.

As above, the equation

$$\mu\Delta\ \mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) = 0$$

has the particular solution $\mathbf{u}_0(x)$ (see (24)) and the last equation has a solution $u = u_0 + V$, where

$$\mu\Delta\ \mathbf{V} + (\lambda+\mu) graddiv\ \mathbf{V} = 0. \tag{34}$$

As it is already clear here $(\mathbf{TV})^+$ is given. Thus we have the second BVP for the equation of an isotropic elastic body. The solution is sought in the form [6]

$$\mathbf{V}(x) = \frac{1}{\pi}\int_S [\mathbf{M}(x,y) - \mathbf{M}(0,y)\mathbf{g}(y)ds, \tag{35}$$

where $\mathbf{g}$ is an unknown function and $\mathbf{M}(x,y)$ has the form

$$\mathbf{M}(x,y) = \frac{1}{2\mu(\lambda+\mu)} Im \begin{pmatrix} ia\ln\sigma - i(\lambda+\mu)\dfrac{\overline{\sigma}}{2\sigma} & -\mu\ln\sigma + (\lambda+\mu)\dfrac{\overline{\sigma}}{2\sigma} \\ \mu\ln\sigma + (\lambda+\mu)\dfrac{\overline{\sigma}}{2\sigma}, & ia\ln\sigma + i(\lambda+\mu)\dfrac{\overline{\sigma}}{2\sigma} \end{pmatrix},$$

where

$$\sigma = x_1 - y_1 + i(x_2 - y_2).$$

From (35), after some operations we find that

$$\mathbf{T}(\partial x, n)\mathbf{V}(x) = \frac{1}{\pi}\int_S \mathbf{T}(\partial x, n)\mathbf{M}(x,y)\mathbf{g}(y)ds, \quad x \in D^+, \tag{36}$$

where

$$\begin{aligned}&\mathbf{T}(\partial x,n)\mathbf{M}(x,y)=\begin{pmatrix}1+\cos 2\theta, & \sin 2\theta\\ \sin 2\theta, & 1-\cos 2\theta\end{pmatrix}\frac{\partial\theta}{\partial s},\\ &\theta=\arctan\frac{y_2-x_2}{y_1-x_1},\quad \frac{\partial \ln r}{\partial n}=\frac{\partial\theta}{\partial s}.\end{aligned}\tag{37}$$

When $x\to z\in S$, for determining the vector $\mathbf{g}$ we obtain the following integral equation

$$-\mathbf{g}(z)+\frac{1}{\pi}\int\limits_S\begin{pmatrix}1+\cos 2\theta, & \sin 2\theta\\ \sin 2\theta, & 1-\cos 2\theta\end{pmatrix}\frac{\partial\theta}{\partial s}\mathbf{g}ds=\mathbf{f}^+(z).\tag{38}$$

The homogeneous equation, corresponding to the equation (38) has nontrivial solution. It is expedient to modify the preceding equation. Therefore we consider the following equation

$$\begin{aligned}&-\mathbf{g}(z)+\frac{1}{\pi}\int\limits_S\mathbf{T}_z\mathbf{M}(z,y)g(y)ds+\frac{1}{2\pi}\mathbf{T}_z\mathbf{M}(z)\int\limits_S\mathbf{g}(y)ds-\\ &\frac{1}{2\pi}\frac{d}{d\psi}\begin{pmatrix}-\sin\psi\sin 2\psi\\ -2\sin^3\psi\end{pmatrix}M=\mathbf{f}^+(z),\quad z\in S,\end{aligned}\tag{39}$$

$$\psi=\arctan\frac{x_2}{x_1},\quad M=\left(\frac{\partial V_2}{\partial x_1}-\frac{\partial V_1}{\partial x_2}\right)_{x_1=x_2=0}.\tag{40}$$

Performing elementary calculation, from (39) we get

$$\int\limits_S\mathbf{g}(y)ds=\int\limits_S\mathbf{f}^+ds,\quad M=\int\limits_S[x_1f_2^+-x_2f_1^+]ds.\tag{41}$$

If the principal vector $\int\limits_S\mathbf{f}^+(y)ds$ and the principal moment $\int\limits_S(x_1f_2^+-x_2f_1^+)ds$ are equal to zero, then $\int\limits_S\mathbf{g}ds=0$ and $M=0$. Then every solution $\mathbf{g}$ of the equation (39) is, at the same time, a solution of the integral equation (38).

Let us prove that the equation (39) is always solvable if the the principal vector and the principal moment are equal to zero. To this end we consider the homogeneous equation obtained from (39) for $\mathbf{f}^+=0$ and prove that it has only trivial solution. Let $\mathbf{g}_0$ be any solution of that equation. Since $\mathbf{f}^+=0$ it is obvious that $\int\limits_S\mathbf{f}^+ds=0$, $M=0$ are fulfilled for $\mathbf{g}_0$. In this case the obtained homogeneous equation corresponds to the boundary condition $(\mathbf{Tu}_0)^+=0$, where $\mathbf{u}_0$ is obtained from (35), if instead of $\mathbf{g}$ we take $\mathbf{g}_0$. Using the uniqueness theorem for the second BVP for D^+, we obtain

$$\mathbf{u}_0(x)=\begin{pmatrix}c_1\\ c_2\end{pmatrix}+\varepsilon\begin{pmatrix}-x_2\\ x_1\end{pmatrix},\quad x\in D^+,$$

where c_j, and ε are arbitrary constants.

Noting that $M_0=0$ and $\mathbf{V}(0)=0$, therefore $\mathbf{u}_0(x)=0$, whence [6] ($\mathbf{u}_0$ and $\mathbf{W}_0$ are the conjugate vectors, $\boldsymbol{\phi}=\boldsymbol{u}_0+i\boldsymbol{W}_0$)

$$0=\mathbf{N}(\partial x,n)\mathbf{u}_0(x)=\frac{\lambda+3\mu}{2a\mu}\frac{\partial\mathbf{W}_0}{\partial S(x)},\quad x\in D^+.\tag{42}$$

From here $\mathbf{W}_0 = c, \ \ x \in D^+$, where [6]

$$\mathbf{W}_0 = \frac{1}{\pi} Re \int\limits_S \begin{pmatrix} \ln\sigma - \dfrac{\lambda+\mu}{\lambda+3\mu}\dfrac{\overline{\sigma}}{2\sigma} & -\dfrac{\lambda+\mu}{\lambda+3\mu}\dfrac{\overline{\sigma}}{2\sigma} \\ -\dfrac{\lambda+\mu}{\lambda+3\mu}\dfrac{\overline{\sigma}}{2\sigma}, & \ln\sigma - \dfrac{\lambda+\mu}{\lambda+3\mu}\dfrac{\overline{\sigma}}{2\sigma} \end{pmatrix} \mathbf{g}(y)ds. \tag{43}$$

We can easily establish that if $\mathbf{g}_0$ is a continuous vector, then $(\mathbf{TW}_0)^+ - (\mathbf{TW}_0)^- = 0$. But since $(\mathbf{TW}_0)^+ = 0$, from the last formula we obtain $(\mathbf{TW}_0)^- = 0$. By virtue of $\int\limits_S \mathbf{g}ds = 0$, the vector $\mathbf{W}_0$ is one valued on the entire plane and of order $|x|^{-1}$ at infinity, $\mathbf{W}(\infty) = 0$. Using this fact and uniqueness theorem we obtain

$$\mathbf{W}_0(x) = 0, \quad x \in D^-. \tag{44}$$

The formula $\mathbf{W}_0 = c, \ \ x \in D^+$ and (44) yield $(\mathbf{L}(\partial x, n)\mathbf{W}_0)^+ = 0, \ \ x \in D^+, \ \ (\mathbf{L}(\partial x, n)\mathbf{W}_0)^- = 0, \ \ x \in D^-$, where the operator $\mathbf{L}(\partial x, n)$ is obtained from $\overset{\kappa}{\mathbf{T}}(\partial x, n)$ for $\kappa = 2\mu$. Further, if we use the formula [6]

$$0 = (\mathbf{L}(\partial x, n)\mathbf{W}_0(x))^+ - (\mathbf{L}(\partial x, n)\mathbf{W}_0(x))^- = \frac{2\mu}{a}\mathbf{g}_0(z),$$

we obtain $\mathbf{g}_0 = 0$.

Thus the homogeneous equation corresponding to the (39) has only trivial solution. Consequently, the equation (39) has a unique solution $\mathbf{g}$. Substituting $\mathbf{g}$ in (35), we get solution of the second BVP, provided the principal vector and the principal moment of external stresses are equal to zero.

Theorem 9. *If $S \in C^{2,\beta}, \ \ 0 < \beta < 1, \ \ f, f_j \in C^{1,\alpha}(S), \ \ 0 < \alpha < \beta$, then a regular solution of problem $(II)^+_f$ exists, it is unique and represented by the potentials of singlelayer (31) and (35), where $\mathbf{g}$ is a solution of the Fredholm integral equations (32) and (39) respectively which are always solvable for an arbitrary right hand side.*

Problem $(II)^-_f$. Now let us prove the existence of solution of the second BVP $((\mathbf{TV})^- = \mathbf{f}^-)$ in the domain D^-. The solution is sought in the form

$$\mathbf{V}(x) = \frac{1}{\pi}\int\limits_S \mathbf{M}(z, y)\mathbf{g}(y)ds + \frac{\mu}{a\rho}\begin{pmatrix} \cos\psi\cos 2\psi \\ \cos\psi \end{pmatrix} M, \tag{45}$$

where

$$\psi = \arctan\frac{x_2}{x_1}, \ \ M = \left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right)_{x_1 = x_2 = 0}. \tag{46}$$

Here $\mathbf{g}$ is an unknown vector. For the vector $\mathbf{V}$ to be single valued and bounded at infinity, we assume that the condition $\int\limits_S \mathbf{g}ds = 0$, is fulfilled. Later on the principal vector will be assumed to be equal to zero.

For determining of vector $\mathbf{g}$ we obtain the following Fredholm integral equation of the second kind

$$\mathbf{g}(z) + \frac{1}{\pi}\int\limits_S \begin{pmatrix} 1 + \cos 2\theta, & \sin 2\theta \\ \sin 2\theta, & 1 - \cos 2\theta \end{pmatrix} \frac{\partial\theta}{\partial s}\mathbf{g}(y)ds + \frac{\mu}{a}\frac{\partial}{\partial\psi}\begin{pmatrix} \cos\psi\cos 2\psi \\ \cos\psi \end{pmatrix} M = \mathbf{f}^-(z). \tag{47}$$

By integration, from (47) we obtain

$$\int_S \mathbf{g} ds = \int_S \mathbf{f}^- ds. \tag{48}$$

Now we will establish that the equation (47) is always solvable. To this end, we consider the homogeneous equation obtained from (47) for $\mathbf{f}^- = 0$. Let's prove that this equation has only trivial solution. Let's assume the contrary and denote by $\mathbf{g}_0$ any solution of the homogeneous equation. Since $\mathbf{f}^- = 0$, from (48) we have $\int_S \mathbf{g}_0 ds = 0$. Note that the homogeneous equation corresponds now to the boundary condition $(\mathbf{TV})^- = 0$. Taking into account the uniqueness theorem for the second BVP in the domain D^-, we obtain $\mathbf{V}_0(x) = 0, x \in D^-$. In this case $(\mathbf{LV}_0)^- = (\mathbf{LV}_0)^+ = 0$. Therefore

$$0 = \int_S [x_1(\mathbf{LV}_0)_1^+ + x_2(\mathbf{LV}_0)_2^-] ds = M_0, \quad x \in D^-$$

and (45) takes the form

$$\mathbf{V}_0 = u_0(x) = \frac{1}{\pi} \int_S \mathbf{M}(x,y) \mathbf{g} ds = 0, x \in D^-.$$

From here

$$0 = \mathbf{N}(\partial x, n) \mathbf{u}_0 = \frac{\lambda + 3\mu}{2a\mu} \frac{\partial \mathbf{W}_0}{\partial s(x)}.$$

The last equation gives $\mathbf{W}_0 = c, \quad x \in D^-$. As since $\mathbf{W}_0(\infty) = 0$, we obtain $c = 0$ and $\mathbf{W}_0 = 0, \quad x \in D^-$. From here it follows that $(\mathbf{TW}_0)^- = 0$. But $(\mathbf{TW}_0)^- = (\mathbf{TW}_0)^+$. Therefore $(\mathbf{TW}_0)^+ = 0$ and

$$\mathbf{W}_0(x) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \varepsilon \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad x \in D^+.$$

By appling (46) we obtain $M_0 = \varepsilon = 0$ and $\mathbf{W}_0 = c, \quad x \in D^+$.

Later having used the formula

$$0 = (\mathbf{L}(\partial x, n) \mathbf{W}_0(x))^+ - (\mathbf{L}(\partial x, n) \mathbf{W}_0(x))^- = \frac{2\mu}{a} \mathbf{g}_0(z).$$

we obtain $\mathbf{g}_0 = 0$.

Consequently (47) has a unique solution, provided the principal vector is equal to zero.

Remark. As above the solution of BVP $\left[\dfrac{\partial p_1}{\partial n}\right]^- = f_3^-(z), \left[\dfrac{\partial p_2}{\partial n}\right]^- = f_4^-(z),$ will be represented by the singlelayer potential (31), where g_3 and g_4 are the solutions of Fredholm integral equations of the second kind

$$-m_2 g_3(z) + \frac{\partial p_1(z)}{\partial n} = f_3(z), \quad -m_1 g_4(z) + \frac{\partial p_2(z)}{\partial n} = f_4(z), \quad z \in S. \tag{49}$$

Thus the existence of the solution of the second boundary value problem in the domain D^- is proved.

Theorem 10. *If* $S \in C^{2,\beta}$, $\quad 0 < \beta < 1, \quad f_3, f_4, \boldsymbol{f} \in C^{1,\alpha}(S), \quad 0 < \alpha < \beta$, *then a regular solution of problem* $(II)^-_f$ *exists, it is unique and represented by the potentials of singlelayer (45) and (31), where* $\boldsymbol{g}$ *is a solution of the Fredholm integral equations (47) and (49) respectively which are always solvable for an arbitrary right hand side.*

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