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SOLUTION OF THE THIRD AND FOURTH BVPs OF THE THEORY OF CONSOLIDATION WITH DOUBLE POROSITY FOR THE SPHERE AND FOR SPACE WITH A SPHERICAL CAVE

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Abstract. The purpose of this paper is to explicitly solve the basic third and the fourth boundary value problems (BVPs) of the theory of consolidation with double porosity for the sphere and for the whole space with a spherical cavity. The obtained solutions are represented as absolutely and uniformly convergent series.

Keywords and phrases: Porous media, double porosity, absolutely and uniformly convergent series, spherical harmonic.

AMS subject classification (2000): 74G05; 74G10.

Introduction

A theory of consolidation with double porosity has been proposed by Aifantis. This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example, in a fissured rock (i.e., a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or the secondary porosity. When fluid flows and deformation processes occur simultaneously , three coupled partial differential equations can be derived [1],[2] to describe the relationships governing pressure in the primary and secondary pores (and therefore the mass exchange between them) and the displacement of the solid.

The physical and mathematical foundations of the theory of double porosity were considered in the papers [1]-[3]. In part I of a series of paper on the subject, R. K. Wilson and E. C. Aifantis [1] gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory.They also solved several representative boundary value problems. In part II of these series, uniqueness and variational principles were established by D. E. Beskos and E. C. Aifantis [2] for the equations of double porosity,while in part III Khaled, Beskos and Aifantis [3] provided a related finite element to consider the numerical solution of Aifantis' equations of double porosity (see [1],[2],[3] and references cited therein). The basic results and the historical information on the theory of porous media were summarized by de Boer [4].

The main goal of this investigation is to construct explicitly, in the form of absolutely and uniformly convergent series, the solutions of the basic the third and the fourth boundary value problems (BVPs) of the theory of consolidation with double porosity for the sphere and for the whole space with spherical cave.

1. Formulation of boundary value problems and uniqueness theorems

The basic Aifantis' equations of statics of the theory of consolidation with double porosity are given in the form [1], [2]

$$\mu\Delta u + (\lambda+\mu) graddivu - grad(\beta_1 p_1 + \beta_2 p_2) = 0, \tag{1.1}$$

$$(m_1\Delta - k)p_1 + kp_2 = 0, \quad kp_1 + (m_2\Delta - k)p_2 = 0, \tag{1.2}$$

where $u = (u_1, u_2, u_3)$ is the displacement vector, p_1 is the fluid pressure within the primary pores and p_2 is the fluid pressure within the secondary pores. . The constant λ is the Lame modulus, μ is the shear modulus and the constants β_1 and β_2 measure the change of porosities due to an applied volumetric strain. $m_j = \dfrac{k_j}{\mu^*}, j = 1, 2.$ The constants k_1 and k_2 are the permeabilities of the primary and secondary systems of pores, the constant μ^* denotes the viscosity of the pore fluid and the constant k measures the transfer of fluid from the secondary pores to the primary pores. The quantities λ, μ, k, β_j, k_j $(j = 1, 2)$ and μ^* are all positive constants. $\triangle$ is Laplace operator.

Let $D^+ = \{x \in E_3 ||x| < a\}$ be an open sphere of radius a centered at point 0 in space E_3 and let $S = \{x \in E_3||x| = a\}$ be a spherical surface of radius a. Denote by D^--whole space with a spherical cave.

Introduce the definition of a regular vector-function.

Definition 1. A vector-function $U(x) = (u_1, u_2, u_3, p_1, p_2)$ defined in the domain $D^+(D^-)$ is called regular if it has integrable continuous second derivatives in $D^+(D^-)$, and U itself and its first order derivatives are continuously extendable at every point of the boundary of $D^+(D^-)$, i.e., $U \in C^2(D^+)\bigcap C^1(\overline{D^+})$, $(U \in C^2(D^-)\bigcap C^1(\overline{D^-}))$. Note that for the infinite domain D^- the vector $U(x)$ additionally satisfies the following conditions at infinity:

$$U(x) = O(|x|^{-1}), \quad \frac{\partial U_k}{\partial x_j} = O(|x|^{-2}), \quad |x|^2 = x_1^2 + x_2^2 + x_2^3, \;\; j = 1, 2, 3. \tag{1.3}$$

For the equations (1.1)-(1.2) we pose the following boundary value problems:

The third internal and external problem (**Problem** $(III)^{\pm}$). Find in $D^+(D^-)$ a regular solution U, of the equations (1.1)-(1.2), by the boundary conditions

$$u^{\pm}(z) = f(z)^{\pm}, \left(\frac{\partial p_1(z)}{\partial n}\right)^{\pm} = f_4^{\pm}, \quad \left(\frac{\partial p_2(z)}{\partial n}\right)^{\pm} = f_5^{\pm}(z), \;\; z \in S,$$

where

$$f^{\pm} \in C^{1,\alpha}(S), \quad f_k^{\pm} \in C^{0,\alpha}(S), \;\; 0 < \alpha \leq 1, \;\; k = 4, 5,$$

are given functions.

The fourth internal and external problem (**Problem** $(IV)^{\pm}$).

Find in $D^+(D^-)$ a regular solution U, of the equations (1.1)-(1.2), by the boundary conditions

$$(Pu)^{\pm} = f(z)^{\pm}, \qquad p_1^{\pm}(z) = f_4^{\pm}, \;\; p_2^{\pm}(z) = f_5^{\pm}(z), \quad z \in S,$$

where $f^{\pm} \in C^{0,\alpha}(S)$, $\;f_k^{\pm} \in C^{1,\alpha}(S)$, $\;0 < \alpha \leq 1$, $\;k = 4, 5$, are given functions, Pu is a stress vector, which acts on an elements of the S with the normal $n = (n_1, n_2, n_3)$

$$P(\partial x, n)u = T(\partial x, n)u - n(\beta_1 p_1 + \beta_2 p_2), \tag{1.4}$$

here $T(\partial x, n)$ is a stress tensor [7]

$$\begin{aligned} &T(\partial x, n) = \| \, T_{kj}(\partial x, n) \, \|_{3x3}, \\ &T_{kj}(\partial x, n) = \mu \delta_{kj} \frac{\partial}{\partial n} + \lambda n_k \frac{\partial}{\partial x_j} + \mu n_j \frac{\partial}{\partial x_k}, \quad k, j, = 1, 2, 3. \end{aligned} \tag{1.5}$$

Further we assume that p_j is known, when $x \in D^+$ or $x \in D^-$. Substitute $\beta_1 p_1 + \beta_2 p_2$ in (1.1) and search the particular solution of the following equation

$$\mu \Delta u + (\lambda + \mu) graddivu = grad(\beta_1 p_1 + \beta_2 p_2).$$

It is known, that a particular solution of the equation (1.1) is the following potential [7]

$$u_0(x) = -\frac{1}{4\pi} \iiint\limits_D \Gamma(x - y) grad(\beta_1 p_1 + \beta_2 p_2) dy, \tag{1.6}$$

where

$$\Gamma(x - y) = \frac{1}{4\mu(\lambda + 2\mu)} \left\| \frac{(\lambda + 3\mu)\delta_{kj}}{r} + \frac{(\lambda + \mu)(x_k - y_k)(x_j - y_j)}{r^3} \right\|_{3\times 3},$$

$$r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2.$$

Substituting the volume potential u_0 into (1.1) we obtain (see [7])

$$\mu \Delta u_0 + (\lambda + \mu) graddivu_0 = grad(\beta_1 p_1 + \beta_2 p_2).$$

Thus we have proved that $u_0(x)$ is a particular solution of the equation (1.1). In (1.6) D denotes either D^+ or D^-, $grad(\beta_1 p_1 + \beta_2 p_2)$ is a continuous vector in D^+ along with its first derivatives. When $D = D^-$ the vector $grad(\beta_1 p_1 + \beta_2 p_2)$ has to satisfy the following condition at infinity

$$grad(\beta_1 p_1 + \beta_2 p_2) = O(|x|^{-2-\alpha}), \alpha > 0.$$

Thus the general solution of the equation (1.1) is representable in the form $u = V + u_0$, where

$$A(\partial x)V = \mu \Delta V + (\lambda + \mu) graddivV = 0. \tag{1.7}$$

The latter equation is the equation of an isotropic elastic body. i.e. we reduce the solution of basic BVPs of the theory of consolidation with double porosity to the solution of the basic BVPs for the equation of an isotropic elastic body.

2. Some auxiliary formulas

The spherical coordinates are defined by the equalities

$$\begin{aligned} &x_1 = \rho \sin\vartheta \cos\varphi, \quad x_2 = \rho \sin\vartheta \sin\varphi, \quad x_3 = \rho\cos\vartheta, \quad x \in D^+, \\ &y_1 = a \sin\vartheta_0 \cos\varphi_0, \quad y_2 = a\sin\vartheta_0 \sin\varphi_0, \quad y_3 = a\cos\vartheta_0, \quad y \in S, \\ &\rho^2 = x_1^2 + x_2^2 + x_3^2, \quad 0 \le \vartheta \le \pi, \quad 0 \le \varphi \le 2\pi, \end{aligned} \tag{2.1}$$

Let

$$f(z) = \sum_{m=0}^{\infty} f_m(\vartheta, \varphi),$$

where f_m is the sperical function of order m :

$$f_m(\vartheta,\varphi) = \frac{2m+1}{4\pi a^2} \int\limits_S P_m(\cos\gamma) f(y) dS_y,$$

P_m is Legender polynomial of the m-th order, γ is an angle formed by the radius-vector Ox and Oy,

$$\cos\gamma = \frac{1}{|x||y|} \sum_{m=1}^{3} x_k y_k.$$

The general solutions of the equation $(\Delta - \lambda_0^2)\psi = 0$ in the domains $D^+(D^-)$ have the form ([6])

$$\begin{aligned} \psi(x) &= \sum_{n=0}^{\infty} \frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}} Y_n(\vartheta,\varphi), \quad \rho < a, \\ \psi(x) &= \sum_{n=0}^{\infty} \frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}} Y_n(\vartheta,\varphi), \quad \rho > a, \end{aligned} \tag{2.2}$$

$$\lambda_0^2 = \frac{k}{m_1} + \frac{k}{m_2} > 0.$$

$Y_n(\vartheta,\varphi)$ is the spherical harmonic.

The general solutions of the equation $\Delta\phi = 0$ in the domains $D^+(D^-)$ have the form ([5], p.505)

$$\begin{aligned} \phi(x) &= \sum_{n=0}^{\infty} \frac{\rho^n}{(2n+1)a^{n-1}} Z_n(\vartheta,\varphi), \quad \rho < a, \\ \phi(x) &= \sum_{n=0}^{\infty} \frac{a^{n+2}}{(2n+1)\rho^{n+1}} Z_n(\vartheta,\varphi), \quad \rho > a, \end{aligned} \tag{2.3}$$

$Z_n(\vartheta,\varphi)$ is the spherical harmonic.

It is easy to show that the general solution of the equation (1.2) is representable in the form

$$p_1 = -m_2\psi + \phi, \quad p_2 = m_1\psi + \phi \tag{2.4}$$

where ψ and ϕ are arbitrary solutions of the following equations

$$(\Delta-\lambda_0^2)\psi=0, \quad \Delta\phi=0.$$

The following theorems are valid and we cite them without proof.

Theorem 1. *The boundary value problems* $(III)^-$, $(IV)^-$ *have at most one regular solution in the domain* D^-.

Theorem 2. *Two regular solutions of the boundary value problem* $(III)^+$ *in the domain* D^+ *may differ by the vector* $V(u,p_1,p_2)$, *where* $u=0$, *and* $p_1=p_2=c$.

Theorem 3. *Two regular solutions of the boundary value problem* $(IY)^+$ *may differ by the vector* $V(u,p_1,p_2)$, *where* u *vector is a rigid displacement* $u_1=c_1-\epsilon x_2$, $u_2=c_2+\epsilon x_1$, *and* $p_1=p_2=0$, $x\in D^+$, ϵ *and* c_j, $j=1,2$, *are arbitrary real constants.*

3. Solution of the third boundary value problem

Problem $(III)^+$. First of all we construct a solution for the equations (1.2). A solution of the boundary value problem ($[\frac{\partial p_1}{\partial n}]^+=f_4^+(z)$, $[\frac{\partial p_2}{\partial n}]^+=f_5^+(z)$) we seek in the following form

$$\begin{aligned}
p_1&=-m_2\sum_{n=0}^{\infty}\frac{J_{n+\frac12}(i\lambda_0\rho)}{\sqrt{\rho}}Y_n(\vartheta,\varphi)+\sum_{n=0}^{\infty}\frac{\rho^n}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi), \quad \rho<a,\\
p_2&=m_1\sum_{n=0}^{\infty}\frac{J_{n+\frac12}(i\lambda_0\rho)}{\sqrt{\rho}}Y_n(\vartheta,\varphi)+\sum_{n=0}^{\infty}\frac{\rho^n}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi)), \quad \rho<a.
\end{aligned} \tag{3.1}$$

Taking into account the fact that $\dfrac{\partial}{\partial n}=\dfrac{\partial}{\partial \rho}$, from the last equation we obtain

$$\begin{aligned}
\frac{\partial p_1}{\partial n}&=\frac{\partial p_1}{\partial \rho}=-m_2\sum_{n=0}^{\infty}\frac{\partial}{\partial\rho}\frac{J_{n+\frac12}(i\lambda_0\rho)}{\sqrt{\rho}}Y_n(\vartheta,\varphi)+\sum_{n=0}^{\infty}\frac{n\rho^{n-1}}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi), \quad \rho<a,\\
\frac{\partial p_2}{\partial n}&=\frac{\partial p_2}{\partial \rho}=m_1\sum_{n=0}^{\infty}\frac{\partial}{\partial\rho}\frac{J_{n+\frac12}(i\lambda_0\rho)}{\sqrt{\rho}}Y_n(\vartheta,\varphi)+\sum_{n=0}^{\infty}\frac{n\rho^{n-1}}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi), \quad \rho<a.
\end{aligned} \tag{3.2}$$

Let us rewrite (3.2) as

$$\begin{aligned}
\frac{\partial p_1}{\partial \rho}&=-m_2\sum_{n=0}^{\infty}H_n(\rho)Y_n(\vartheta,\varphi)+\sum_{n=0}^{\infty}\frac{n\rho^{n-1}}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi)), \quad \rho<a,\\
\frac{\partial p_2}{\partial \rho}&=m_1\sum_{n=0}^{\infty}H_n(\rho)Y_n(\vartheta,\varphi)+\sum_{n=0}^{\infty}\frac{n\rho^{n-1}}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi), \quad \rho<a,
\end{aligned} \tag{3.3}$$

where $H_n(\rho)=\dfrac{\partial}{\partial\rho}\dfrac{J_{n+\frac12}(i\lambda_0\rho)}{\sqrt{\rho}}$.

Let

$$f_k(z)=\sum_{n=0}^{\infty}\widehat{f}_{nk}(\vartheta_0,\varphi_0),$$

where $\widehat{f}_{nk}, \quad k=4,5$ is the sperical function of order n :

$$\widehat{f}_{nk}(\vartheta_0,\varphi_0)=\frac{2n+1}{4\pi a^2}\int\limits_S P_n(\cos\gamma)f_k(y)dS_y, \quad k=4,5.$$

Passing to the limit in (3.3) as $D^+\ni\rho\to a$, we obtain

$$\begin{aligned}&-m_2\sum_{n=0}^{\infty}H_n(a)Y_n(\vartheta_0,\varphi_0)+\sum_{n=0}^{\infty}\frac{n}{(2n+1)}Z_n(\vartheta_0,\varphi_0)=\sum_{n=0}^{\infty}\widehat{f}_{4n}(\vartheta_0,\varphi_0),\\ &m_1\sum_{n=0}^{\infty}H_n(a)Y_n(\vartheta_0,\varphi_0)+\sum_{n=0}^{\infty}\frac{n}{(2n+1)}Z_n(\vartheta_0,\varphi_0)=\sum_{n=0}^{\infty}\widehat{f}_{5n}(\vartheta_0,\varphi_0).\end{aligned} \tag{3.4}$$

For the coefficients of Y_n and Z_n, (3.4) yields the following equations:

$$\begin{aligned}&-m_2H_n(a)Y_n(\vartheta_0,\varphi_0)+\frac{n}{(2n+1)}Z_n(\vartheta_0,\varphi_0)=\widehat{f}_{4n}(\vartheta_0,\varphi_0),\\ &m_1H_n(a)Y_n(\vartheta_0,\varphi_0)+\frac{n}{(2n+1)}Z_n(\vartheta_0,\varphi_0)=\widehat{f}_{5n}(\vartheta_0,\varphi_0), \quad n=1,2,..\end{aligned} \tag{3.5}$$

By elementary calculation from (3.5) we define Y_n and Z_n, for $n\geq 1$

$$\begin{aligned}&Y_n(\vartheta_0,\varphi_0)=\frac{\widehat{f}_{5n}(\vartheta_0,\varphi_0)-\widehat{f}_{4n}(\vartheta_0,\varphi_0)}{(m_1+m_2)H_n(a)},\\ &Z_n(\vartheta_0,\varphi_0)=\frac{(2n+1)[m_1\widehat{f}_{4n}(\vartheta_0,\varphi_0)+m_2\widehat{f}_{5n}(\vartheta_0,\varphi_0)]}{n(m_1+m_2)}, \quad n=1,2,...\end{aligned} \tag{3.6}$$

Note that Z_0 is an arbitrary constant and

$$Y_0=\int\limits_S f_4dS=\int\limits_S f_5dS=0.$$

Substituting (3.6) into (3.1), we obtain a solution of the BVP in the form of series

$$\begin{aligned}p_1=&\frac{-m_2}{(m_1+m_2)\sqrt{\rho}}\sum_{n=1}^{\infty}\frac{J_{n+\frac12}(i\lambda_0\rho)}{H_n(a)}[\widehat{f}_{5n}(\vartheta,\varphi)-\widehat{f}_{4n}(\vartheta,\varphi)]\\ &+\frac{1}{m_1+m_2}\sum_{n=1}^{\infty}\frac{\rho^n}{na^{n-1}}[m_1\widehat{f}_{4n}(\vartheta,\varphi)+m_2\widehat{f}_{5n}(\vartheta,\varphi)]+c, \quad \rho<a,\\ p_2=&\frac{m_1}{(m_1+m_2)\sqrt{\rho}}\sum_{n=1}^{\infty}\frac{J_{n+\frac12}(i\lambda_0\rho)}{H_n(a)}[\widehat{f}_{5n}(\vartheta,\varphi)-\widehat{f}_{4n}(\vartheta,\varphi)]\\ &+\frac{1}{m_1+m_2}\sum_{n=1}^{\infty}\frac{\rho^n}{na^{n-1}}[m_1\widehat{f}_{4n}(\vartheta,\varphi)+m_2\widehat{f}_{5n}(\vartheta,\varphi)]+c, \quad \rho<a.\end{aligned} \tag{3.7}$$

Problem $(III)^-$. The boundary value problem $[\frac{\partial p_1}{\partial n}]^- = f_4^-(z)$, $[\frac{\partial p_2}{\partial n}]^- = f_5^-(z)$ can be solved analogously and we have

$$\begin{aligned} p_1 = {} & \frac{-m_2}{(m_1+m_2)\sqrt{\rho}} \sum_{n=1}^{\infty} \frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{h_n(a)} [\widehat{f}_{5n}(\vartheta,\varphi) - \widehat{f}_{4n}(\vartheta,\varphi)] \\ & - \frac{1}{m_1+m_2} \sum_{n=1}^{\infty} \frac{a^{n+2}}{(n+1)\rho^{n+1}} [m_1 \widehat{f}_{4n}(\vartheta,\varphi) + m_2 \widehat{f}_{5n}(\vartheta,\varphi)], \quad \rho > a, \\ p_2 = {} & \frac{m_1}{(m_1+m_2)\sqrt{\rho}} \sum_{n=1}^{\infty} \frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{h_n(a)} [\widehat{f}_{5n}(\vartheta,\varphi) - \widehat{f}_{4n}(\vartheta,\varphi)] \\ & - \frac{1}{m_1+m_2} \sum_{n=1}^{\infty} \frac{a^{n+2}}{(n+1)\varrho^{n+1}} [m_1 \widehat{f}_{4n}(\vartheta,\varphi) + m_2 \widehat{f}_{5n}(\vartheta,\varphi)], \quad \rho > a, \end{aligned} \tag{3.8}$$

where $h_n(\rho) = \dfrac{\partial}{\partial \rho} \dfrac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}}$.

The functions $\dfrac{\partial p_k}{\partial \rho}$ can be calculated from (3.7)-(3.8).

The solution of the equation

$$\mu \Delta V + (\lambda+\mu) graddiv V = 0,$$

when $V^{\pm} = F^{\pm}$ for a ball is due to Natroshvili D. [8]. (A detailed exposition of the solution can be found in monograph [7]).

$$V(x) = \iint\limits_S \overset{(1)+}{\mathrm{K}}(x,y) F^+(y) d_y S, \quad x \in D^+, \quad y \in S,$$

$$V(x) = \iint\limits_S \overset{(1)-}{\mathrm{K}}(x,y) F^-(y) d_y S, \quad x \in D^-, \quad y \in S,$$

where

$$\overset{(1)+}{\mathrm{K}} = \| \overset{(1)+}{\mathrm{K}}_{kj} \|_{3\times 3},$$

$$\overset{(1)+}{\mathrm{K}}_{kj} = \frac{1}{4\pi a} \left[\frac{a^2-\rho^2}{r^3} \delta_{ij} + \beta (a^2-\rho^2) \frac{\partial^2 \Phi(x,y)}{\partial x_i \partial x_j} \right],$$

$$\Phi(x,y) = \int_0^1 \left[\frac{a^2 - \rho^2 t^2}{Q(t)} - \frac{1}{a} - \frac{3t\rho cos\gamma}{a^2} \right] \frac{dt}{t^{1+\alpha}},$$

$$Q(t) = (a^2 - 2a\rho t cos\gamma + \rho^2 t^2)^{\frac{3}{2}},$$

$$\overset{(1)-}{\mathrm{K}} = \| \overset{(1)-}{\mathrm{K}}_{kj} \|_{3\times 3},$$

$$\overset{(1)-}{\underset{kj}{\mathrm{K}}} = \frac{1}{4\pi a}\left[\frac{\rho^2-a^2}{r^3}\delta_{ij} + \beta(\rho^2-a^2)\frac{\partial^2\Phi^*(x,y)}{\partial x_i \partial x_j}\right],$$

$$\Phi^*(x,y) = \int_0^1 \frac{\rho^2-a^2t^2}{Q^*(t)}t^\alpha dt, \quad Q^*(t) = (\rho^2 - 2a\rho t cos\gamma + a^2t^2)^{\frac{3}{2}},$$

$$cos\gamma = \frac{x_1y_1+x_2y_2+x_3y_3}{ar} = sin\theta sin\theta' cos(\phi-\phi') + cos\theta cos\theta',$$

$$r^2 = a^2 - 2atcos\gamma + \rho^2, \quad \beta = \frac{\lambda+\mu}{(2\lambda+3\mu)}, \quad \alpha = \frac{\lambda+2\mu}{2(\lambda+3\mu)} < 1, \quad F^\pm \in C^{1,\alpha}(S).$$

Finally we have proved the following

Theorem 4. *The third BVP $(III)^-$is uniquely solvable in the class of regular functions and the solution is represented in the form of absolutely and uniformly convergent series if the boundary data are from space $C^{1,\alpha}(S)$, $\alpha > \frac{1}{2}$. The solution of third BVP $(III)^+$ is represented in the form of absolutely and uniformly convergent series if the boundary data are from space $C^{1,\alpha}(S)$, $\alpha > \frac{1}{2}$ and two regular solutions of the boundary value problem $(III)^+$ in the domain D^+ may differ only to within additive constant $c, p_j = c, j = 1, 2$.*

4. Solution of the fourth boundary value problem

Problem $(IV)^+$.First of all we will construct a solution for the equations (1.2). A solution of the boundary value problem ($p_1^+(z) = f_4^+$, $p_2^+(z) = f_5^+(z)$,) is sought in the form (3.1):

Passing to the limit in (3.1) as $D^+ \ni \rho \to a$, we have

$$\begin{aligned} &-m_2\sum_{n=0}^{\infty}\frac{J_{n+\frac{1}{2}}(i\lambda_0 a)}{\sqrt{a}}Y_n(\vartheta_0,\varphi_0) + a\sum_{n=0}^{\infty}\frac{1}{(2n+1)}Z_n(\vartheta_0,\varphi_0) = \sum_{n=0}^{\infty}\widehat{f}_{4n}(\vartheta_0,\varphi_0),\\ &m_1\sum_{n=0}^{\infty}\frac{J_{n+\frac{1}{2}}(i\lambda_0 a)}{\sqrt{a}}Y_n(\vartheta_0,\varphi_0) + a\sum_{n=0}^{\infty}\frac{1}{(2n+1)}Z_n(\vartheta_0,\varphi_0) = \sum_{n=0}^{\infty}\widehat{f}_{5n}(\vartheta_0,\varphi_0), \end{aligned} \tag{4.1}$$

For the coefficients of Y_n and Z_n, (4.1) yields the following equations:

$$\begin{aligned} &-m_2\frac{J_{n+\frac{1}{2}}(i\lambda_0 a)}{\sqrt{a}}Y_n(\vartheta_0,\varphi_0) + \frac{a}{2n+1}Z_n(\vartheta_0,\varphi_0) = \widehat{f}_{4n}(\vartheta_0,\varphi_0),\\ &m_1\frac{J_{n+\frac{1}{2}}(i\lambda_0 a)}{\sqrt{a}}Y_n(\vartheta_0,\varphi_0) + \frac{a}{2n+1}Z_n(\vartheta_0,\varphi_0) = \widehat{f}_{5n}(\vartheta_0,\varphi_0), \end{aligned} \tag{4.2}$$

By elementary calculation from (4.2) we obtain

$$\begin{aligned} Y_n(\vartheta_0,\varphi_0) &= \frac{\widehat{f}_{5n}(\vartheta_0,\varphi_0) - \widehat{f}_{4n}(\vartheta_0,\varphi_0)}{(m_1+m_2)J_{n+\frac{1}{2}}(i\lambda_0 a)}\sqrt{a},\\ Z_n(\vartheta_0,\varphi_0) &= \frac{(2n+1)[m_1\widehat{f}_{4n}(\vartheta_0,\varphi_0) + m_2\widehat{f}_{5n}(\vartheta_0,\varphi_0))]}{a(m_1+m_2)}. \end{aligned} \tag{4.3}$$

Substituting (4.3) into (3.1), we obtain a solution of the BVP in the form of a series

$$p_1 = \frac{-m_2\sqrt{a}}{(m_1+m_2)\sqrt{\rho}}\sum_{n=0}^{\infty}\frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{J_{n+\frac{1}{2}}(i\lambda_0 a)}(\widehat{f}_{5n}(\vartheta,\varphi)-\widehat{f}_{4n}(\vartheta,\varphi))$$
$$+\frac{1}{(m_1+m_2)}\sum_{n=0}^{\infty}\frac{\rho^n}{a^n}[m_1\widehat{f}_{4n}(\vartheta,\varphi)+m_2\widehat{f}_{5n}(\vartheta,\varphi)],$$
$$p_2 = \frac{m_1\sqrt{a}}{(m_1+m_2)\sqrt{\rho}}\sum_{n=0}^{\infty}\frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{J_{n+\frac{1}{2}}(i\lambda_0 a)}(\widehat{f}_{5n}(\vartheta,\varphi)-\widehat{f}_{4n}(\vartheta,\varphi))$$
$$+\frac{1}{(m_1+m_2)}\sum_{n=0}^{\infty}\frac{\rho^n}{a^n}[m_1\widehat{f}_{4n}(\vartheta,\varphi)+m_2\widehat{f}_{5n}(\vartheta,\varphi)],\quad \rho<a,$$

Problem $(IV)^-$. Analogously we construct a solution of the BVP $p_1^-(z) = f_4^-$, $p_2^-(z)=f_5^-(z)$, in the domain D^-

$$p_1 = \frac{-m_2\sqrt{a}}{(m_1+m_2)\sqrt{\rho}}\sum_{n=0}^{\infty}\frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0 a)}[\widehat{f}_{5n}(\vartheta,\varphi)-\widehat{f}_{4n}(\vartheta,\varphi)]$$
$$+\frac{1}{(m_1+m_2)}\sum_{n=0}^{\infty}\frac{a^{n+1}}{\rho^{n+1}}[m_1\widehat{f}_{4n}(\vartheta,\varphi)+m_2\widehat{f}_{5n}(\vartheta,\varphi)],$$
$$p_2 = \frac{m_1\sqrt{a}}{(m_1+m_2)\sqrt{\rho}}\sum_{n=0}^{\infty}\frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0 a)}[\widehat{f}_{5n}(\theta,\phi)-\widehat{f}_{4n}(\theta,\phi)]$$
$$+\frac{1}{(m_1+m_2)}\sum_{n=0}^{\infty}\frac{a^{n+1}}{\rho^{n+1}}[m_1\widehat{f}_{4n}(\vartheta,\varphi)+m_2\widehat{f}_{5n}(\vartheta,\varphi)],\quad \rho>a.$$

For these series together with their first derivatives to be absolutely and uniformly convergent it is sufficient that $f_k^{\pm}\in C^{1,\alpha}(S)$, $0<\alpha\le 1$, $k=4,5$. Solutions obtained under such conditions are regular in D^+.

The solution of the problem $(TV)^{\pm}=F^{\pm}$, for the equation (1.8) for a ball is given in the work by D. Natroshvili [8] (A detailed exposition of the solution can be found in monograph [7]).

$$V(x)=\iint_S \overset{(2)+}{\mathrm{K}}(x,y)F^+(y)d_ys + a_1 + [\omega,x] + \frac{c(\beta_1+\beta_2)}{3\lambda+2\mu}x,\quad x\in D^+,$$
$$TV = \frac{1}{4\pi\rho}\iint_S \|\ \frac{a^2-\rho^2}{r^3}\delta_{ij} + (a^2-\rho^2)\frac{\partial^2\Phi_4(x,y)}{\partial x_i\partial x_j}\ \|_{3x3}\ F^+(y)ds,\quad x\in D^+,$$
$$V(x)=\iint_S \overset{(2)-}{\mathrm{K}}(x,y)F^-(y)d_ys,\quad x\in D^-,$$
$$TV = \frac{1}{4\pi\rho}\iint_S \|\ \frac{\rho^2-a^2}{r^3}\delta_{ij} + (\rho^2-a^2)\frac{\partial^2\Phi_4^*(x,y)}{\partial x_i\partial x_j}\ \|_{3x3}\ F^-(y)ds,\quad x\in D^-,$$

where

$$\overset{(2)+}{\mathrm{K}} = \|\ \overset{(2)+}{\underset{kj}{\mathrm{K}}}\ \|_{3\times 3},$$

$$\overset{(2)+}{\underset{kj}{\mathrm{K}}} = \frac{1}{8\mu\pi}\left[(\Phi_1+\Phi_2)\delta_{ij} + \frac{a^2-3\rho^2}{2}\frac{\partial^2\Phi_3(x,y)}{\partial x_i\partial y_j} + x_j\frac{\partial}{\partial x_i}(\Phi_1-\Phi_2) - 2x_i\frac{\partial\Phi_1}{\partial x_j}\right]$$
$$+\frac{1}{8\mu\pi}\left[x_i\frac{\partial}{\partial x_j}(2\rho\frac{\partial\Phi_3}{\partial\rho}-\Phi_3) + \rho^2\left(\frac{\partial^2\Phi_2(x,y)}{\partial x_i\partial y_j} - \frac{\partial^2\Phi_1(x,y)}{\partial x_i\partial y_j}\right)\right],$$
$$\Phi_1(x,y) = \int_0^1\left[\frac{a^2-\rho^2t^2}{Q(t)} - \frac{1}{a}\right]\frac{dt}{t}, \quad Q(t) = (a^2-2a\rho t cos\gamma+\rho^2t^2)^{\frac{3}{2}},$$
$$\Phi_2(x,y) = \int_0^1\left[\frac{a^2-\rho^2t^2}{Q(t)} - \frac{1}{a} - \frac{3t\rho cos\gamma}{a^2}\right]\frac{dt}{t^2},$$
$$\Phi_0(x,y) = \int_0^1\left[\frac{a^2-\rho^2t^2}{Q(t)} - \frac{1}{a}\right]\frac{dt}{t^{1+\alpha_1}}, \quad \Phi_3 = \frac{1}{b_1}Im\Phi_0, \quad \Phi_4 = Re(b_2\Phi_0),$$
$$\alpha_1 = b_0+ib_1 = \frac{\mu+i\sqrt{2\lambda^2+6\lambda\mu+3\mu^2}}{2(\lambda+\mu)}, \quad b_2 = \frac{1}{2} + \frac{3\lambda+4\mu}{2\sqrt{2\lambda^2+6\lambda\mu+3\mu^2}},$$

$$\overset{(2)-}{\mathrm{K}} = \| \overset{(2)-}{\underset{kj}{\mathrm{K}}} \|_{3\times3},$$

$$\overset{(2)-}{\underset{kj}{\mathrm{K}}} = \frac{1}{8\mu\pi}\left[-(\Phi_1^*+\Phi_2^*)\delta_{ij} + \frac{a^2-3\rho^2}{2}\frac{\partial^2\Phi_3^*(x,y)}{\partial x_i\partial y_j} - x_j\frac{\partial}{\partial x_i}(\Phi_1-\Phi_2) + 2x_i\frac{\partial\Phi_1^*}{\partial x_j}\right]$$
$$+\frac{1}{8\mu\pi}\left[x_i\frac{\partial}{\partial x_j}(2\rho\frac{\partial\Phi_3^*}{\partial\rho}-\Phi_3^*) - \rho\left(\frac{\partial^2\Phi_2(x,y)}{\partial x_i\partial y_j} - \frac{\partial^1\Phi_1(x,y)}{\partial x_i\partial y_j}\right)\right],$$

$$\Phi_l^*(x,y) = \int_0^1\frac{\rho^2-a^2t^2}{Q^*(t)}t^{l-1}dt, \quad l=1,2, \quad \Phi_3^* = \frac{2(\lambda+\mu)}{\sqrt{2\lambda^2+6\lambda\mu+3\mu^2}}Im\int_0^1\frac{\rho^2-a^2t^2}{Q^*(t)}\frac{dt}{t^{\alpha_2}}$$
$$\Phi_4^*(x,y) = ReA\int_0^1\frac{\rho^2-a^2t^2}{Q^*(t)}\frac{dt}{t^{\alpha_2}}, \quad Q^*(t) = (\rho^2-2a\rho t cos\gamma+a^2t^2)^{\frac{3}{2}},$$
$$\alpha_2 = \frac{-\mu+i\sqrt{2\lambda^2+6\lambda\mu+3\mu^2}}{2(\lambda+\mu)}, \quad A = \frac{1}{2} - i\frac{3\lambda+4\mu}{2\sqrt{2\lambda^2+6\lambda\mu+3\mu^2}}.$$

Thus we have proved the following

Theorem 5. *For the solvability of the BVP $(IV)^+$ it is necessary that the principal vector and the principal moment of external forces be equal to zero. The BVP $(IV)^+$ is solvable in the class of regular functions and the solution is represented in the form of absolutely and uniformly convergent series if the boundary data are from space $C^{0,\alpha}(S)$, $\alpha > \frac{1}{2}$. Two regular solutions of BVP $(IV)^+$ may differ only to within additive vector $a+[b,x]$, where a,b, are arbitrary real constant vectors, $x = x(x_1,x_2,x_3)$. The BVP $(IV)^-$ is solvable in the class of regular functions and the solution is represented in the form of absolutely and uniformly convergent series.*

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