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NECESSARY OPTIMALITY CONDITIONS OF SINGULAR CONTROLS IN CONTROL PROBLEM FOR VOLTERRA TYPE TWO-DIMENSIONAL DIFFERENCE EQUATION

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Abstract. Necessary optimality condition is obtained in the form of discrete maximum principle in an optimal control problem described by a system of two-dimensional difference equations of Volterra type. Moreover, the case of degeneration of discrete maximum condition is considered.

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1. Introduction

Optimization problems for Volterra integral equations occupy an important place in the theory of optimal control. The Volterra integral equations are widely used in modeling some phenomena of continuum mechanics and biomechanics [1-8]. The optimal control problems described by Volterra integral equations have been studied in [8-11]. The present paper deals with investigation of an optimal control problem described by system of Volterra two-dimensional difference equations. The necessary optimality condition is proved in the form of discrete maximum condition. Moreover, necessary optimality conditions are proved for controls which are singular optimal controls in the sense of Pontryagin's maximum principle.

2. Statement of the problem

Consider a problem on minimum of the functional

$$S(u) = \varphi(z(t_1, x_1)), \tag{2.1}$$

under restrictions

$$u(t,x) \in U \subset R^r,\ (t,x) \in T \times X = \{(t,x) :\ t = t_0, t_0+1, ..., t_1 \\ ;\ x = x_0, x_0+1, ..., x_1\}, \tag{2.2}$$

$$z(t,x) = \sum_{\tau=t_0}^{t} \sum_{s=x_0}^{x} f(t,x,\tau,s,z(\tau,s),u(\tau,s)), \quad (t,x) \in T \times X \tag{2.3}$$

Here $\varphi(z)$ is a given twice differentiable scalar function, t_0, t_1, x_0, x_1 are given numbers, $f(t,x,\tau,s,z,u)$ is a given n -dimensional vector-function continuous by the aggregate of variables together with partial derivatives with respect to z up to the second order inclusive, $u(t,x)$ is a control function, U is a given non-empty and bounded set.

A control function $u(t,x)$ satisfying the restriction (2.2) and the pair $(u(t,x),\ z(t,x))$ will be called an admissible control and an admissible process, respectively.

Equation (2.3) is a difference analogue of Volterra two-dimensional integral equation. It is assumed that to each admissible control $u(t,x)$ corresponds unique solution of discrete equation (2.2). The existence, uniqueness and boundedness problems of solutions of Volterra one-dimensional difference equations have been investigated in [5, 12-14].

We note that different aspects of multi parameter, in particular two-parameter discrete control systems have been studied in [15-22].

The admissible control $u(t,x)$ minimizing the functional (2.1) under restrictions (2.2), (2.3) is said to be an optimal control, the corresponding process $(u(t,x),\ z(t,x))$ an optimal process.

3. The second order increment formula

In this section we derive representation for the increments of cost functional $S(u)$. Let the set

$$f(t,x,\tau,s,z,U)=\{\alpha:\ \alpha=f(t,x,\tau,s,z,v),\ v\in U\}\,, \tag{3.1}$$

be convex for all (t,x,τ,s,z) .

Let $(u(t,x),\ z(t,x))$ be a fixed admissible process, by $u(t,x;\ \varepsilon)$ we denote an arbitrary admissible control such that its appropriate state of the process $z(t,x;\ \varepsilon)$ satisfies the relation

$$z(t,x;\varepsilon)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}f(t,x,\tau,s,z(\tau,s:\varepsilon),u(\tau,s:\varepsilon))\equiv\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Big[f(t,x,\tau,s,z(\tau,s:\varepsilon)$$
$$,u(\tau,s))+\varepsilon\left[f(t,x,\tau,s,z(\tau,s:\varepsilon),v(\tau,s))-f(t,x,\tau,s,z(\tau,s:\varepsilon),u(\tau,s))\right]\Big], \tag{3.2}$$

where $\varepsilon\in[0,1]$ is an arbitrary number, $v(t,x)\in U,\ (t,x)\in T\times X$ is an arbitrary admissible control.

Such an admissible control $u(t,x;\varepsilon)$ exists by the convexity of set (3.1).

Introduce the functions

$$y(t,x)=\left.\frac{\partial z(t,x:\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0};\quad Y(t,x)=\left.\frac{\partial^2 z(t,x:\varepsilon)}{\partial\varepsilon^2}\right|_{\varepsilon=0}. \tag{3.3}$$

Using (3.2), and taking into account the smoothness of the function $f(t,x,\tau,s,z,u)$, it is proved that $y(t,x)$ and $Y(t,x)$ are the solutions of Volterra type linear inhomogeneous difference equations

$$y(t,x)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\big[f_z(t,x,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)$$
$$+\Delta_{v(\tau,s)}f(t,x,\tau,s,z(\tau,s),u(\tau,s))\big], \tag{3.4}$$

$$Y(t,x)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\big[f_z(t,x,\tau,s,z(\tau,s),u(\tau,s))Y(\tau,s)+2\Delta_{v(\tau,s)}f_z(t,x,\tau,s,z(\tau,s)$$
$$,u(\tau,s))y(\tau,s)+y^{'}(\tau,s)f_{zz}(t,x,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)\big]. \tag{3.5}$$

Here and in the sequel, we use the denotation

$$\Delta_{v(\tau,s)}f(t,x,\tau,s,z(\tau,s),u(\tau,s))\equiv f(t,x,\tau,s,z(\tau,s),v(\tau,s))$$
$$-f(t,x,\tau,s,z(\tau,s),u(\tau,s)),$$

($'$) prime means a scalar product for the vectors, the transpose operation for the matrices. Moreover, special increment of functional (2.1) responding to admissible controls $u(t,x;\varepsilon)$ and $u(t,x)$ will be written in the form

$$\Delta S_\varepsilon(u)=S(u(t,x;\varepsilon))-S(u(t,x))=\varepsilon\frac{\partial\varphi^{'}(z(t_1,x_1))}{\partial z}y(t_1,x_1)$$
$$+\frac{\varepsilon^2}{2}y^{'}(t_1,x_1)\frac{\partial^2\varphi^{'}(z(t_1,x_1))}{\partial z^2}y(t_1,x_1)+\frac{\varepsilon^2}{2}\frac{\partial\varphi(z(t_1,x_1))}{\partial z}Y(t_1,x_1)+0(\varepsilon^2). \tag{3.6}$$

Now we Introduce the Hamilton-Pontryagins function

$$H(t,x,z(t,x),u(t,x),\psi(t,x))=\sum_{\tau=t}^{t_1}\sum_{s=x}^{x_1}\psi^{'}(\tau,s)f(\tau,s,t,x,z(t,x),u(t,x))$$
$$-\varphi_z^{'}(z(t_1,x_1))f(t_1,x_1,t,x,z(t,x),u(t,x)),$$

where $\psi=\psi(t,x)$ is n-dimensional vector-function of conjugated variables being a solution of the equation

$$\psi(t,x)=H_z(t,x,z(t,x),u(t,x),\psi(t,x)). \tag{3.7}$$

Equation (3.7) is an analogy of the conjugated system [23-25] for control problem (2.1)-(2.3) and is a Volterra linear nonhomogeneous equation with respect to $\psi(t,x)$.

Theorem 3.1 *The second order increment of functional (2.1) can be represented by the following formula*

$$\Delta S_\varepsilon(u)=-\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Delta_{v(t,x)}H(t,x,z(t,x),u(t,x),\psi(t,x))+\frac{\varepsilon^2}{2}\Bigg\{y^{'}(t_1,x_1)\frac{\partial^2\varphi(z(t_1,x_1))}{\partial z^2}$$
$$\times y(t_1,x_1)-\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}y^{'}(t,x)H_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$-2\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Delta_{v(t,x)}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)\Bigg\}+0(\varepsilon^2). \tag{3.8}$$

Proof. Multiplying scalarly the both sides of relations (3.4), (3.5) from the left by $\psi(t,x)$, and summing the both sides of the obtained relations over t (x) from t_0 (x_0)

to t_1 (x_1), we get

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi'(t,x)y(t,x)=\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi'(t,x)\Bigg[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Big[f_z(t,x,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)$$
$$+\Delta_{v(\tau,s)}f(t,x,\tau,s,z(\tau,s),u(\tau,s))\Big]\Bigg]. \tag{3.9}$$

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi'(t,x)Y(t,x)=\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi'(t,x)\Bigg[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Big[f_z(t,x,\tau,s,z(\tau,s),u(\tau,s))Y(\tau,s)$$
$$+2\Delta_{v(\tau,s)}f_z(t,x,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)$$
$$+y'(\tau,s)f_{zz}(t,x,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)\Big]\Bigg]. \tag{3.10}$$

The following statement is true.

Lemma 3.1 *Let $L(t,x,\tau,s)$ and $K(t,x,\tau,s)$ be given $(n\times n)$ matrix functions. Then the identity*

$$\sum_{t=t_0}^{m}\sum_{x=x_0}^{\ell}\left[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}L(m,\ell,t,x)K(t,x,\tau,s)\right]=\sum_{t=t_0}^{m}\sum_{x=x_0}^{\ell}\left[\sum_{\tau=t}^{m}\sum_{s=x}^{\ell}L(m,\ell,\tau,s)K(\tau,s,t,x)\right]$$

is valid.

The lemma is a two-dimensional discrete analogue of Fubini formula [1, 7]. Using this lemma and assuming

$$M(t,x,z(t,x),u(t,x),\psi(t,x))=\sum_{\tau=t}^{t_1}\sum_{s=x}^{x_1}\psi'(\tau,s)f(\tau,s,t,x,z(t,x),u(t,x)),$$

identities (3.9), (3.10) can be transformed into the form

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi'(t,x)y(t,x)=\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[M_z'(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$+\Delta_{v(t,x)}M(t,x,z(t,x),u(t,x),\psi(t,x))\Big], \tag{3.11}$$

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi'(t,x)Y(t,x)=\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[M_z(t,x,z(t,x),u(t,x),\psi(t,x))Y(t,x)$$
$$+2\Delta_{v(t,x)}M_z'(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$+y'(t,x)M_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)\Big]. \tag{3.12}$$

Further, it is clear that from (3.4), (3.5) follows

$$y(t_1,x_1)=\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[f_z(t_1,x_1,t,x,z(t,x),u(t,x))y(t,x)$$

$$+\Delta_{v(t,x)}f(t_1,x_1,t,x,z(t,x),u(t,x))\Big].$$

$$Y(t_1,x_1)=\sum_{\tau=t_0}^{t_1}\sum_{s=x_0}^{x_1}\Big[f_z(t_1,x_1,\tau,s,z(\tau,s),u(\tau,s))Y(\tau,s)$$
$$+2\Delta_{v(\tau,s)}f_z(t_1,x_1,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)$$
$$+y^{'}(\tau,s)f_{zz}(t_1,x_1,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)\Big].$$

Taking into account identities (3.11)-(3.13) in (3.6), we get

$$\Delta S_\varepsilon(u)=\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\frac{\partial\varphi^{'}(z(t_1,x_1))}{\partial z}\Big[f_z(t_1,x_1,t,x,z(t,x),u(t,x))y(t,x)$$
$$+\Delta_{v(t,x)}f(t_1,x_1,t,x,z(t,x),u(t,x))\Big]+\frac{\varepsilon^2}{2}y^{'}(t_1,x_1)\frac{\partial^2\varphi^{'}(z(t_1,x_1))}{\partial z^2}y(t_1,x_1)$$
$$+\frac{\varepsilon^2}{2}\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\frac{\partial\varphi(z(t_1,x_1))}{\partial z}\big[f_z(t_1,x_1,t,x,z(t,x),u(t,x))Y(t,x)$$
$$+2\Delta_{v(\tau,s)}f_z(t_1,x_1,t,x,z(t,x),u(t,x))y(t,x)$$
$$+y^{'}(t,x)f_{zz}(t_1,x_1,t,x,z(t,x),u(t,x))\big]+\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)\,y(t,x)$$
$$-\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[M_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)+\Delta_{v(t,x)}M(t,x,z(t,x),u(t,x),\psi(t,x))\Big]$$
$$+\frac{\varepsilon^2}{2}\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)\,Y(t,x)-\frac{\varepsilon^2}{2}\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[M_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))Y(t,x)$$
$$+2\Delta_{v(t,x)}M_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$+y^{'}(t,x)M_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)\big]+0(\varepsilon^2).$$

Hence, grouping the similar terms and taking into consideration the expressions of Hamilton-Pontryagins function, we have

$$\Delta S_\varepsilon(u)=-\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$-\frac{\varepsilon^2}{2}\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))Y(t,x)+\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)\,y(t,x)$$
$$+\frac{\varepsilon^2}{2}\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)\,Y(t,x)-\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Delta_{v(t,x)}H^{'}(t,x,z(t,x),u(t,x),\psi(t,x))$$
$$+\frac{\varepsilon^2}{2}y^{'}(t_1,x_1)\frac{\partial^2\varphi^{'}(z(t_1,x_1))}{\partial z^2}y(t_1,x_1)-\varepsilon^2\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Delta_{v(t,x)}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))$$
$$\times y(t,x)-\frac{\varepsilon^2}{2}\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}y^{'}(t,x)H_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)+0(\varepsilon^2).$$

Hence, with regard to the fact that $\psi(t,x)$ is a solution of equation (3.7), we obtain formula (3.8).

4. The second order increment formula

By arbitrariness of $\varepsilon \in [0,1]$ the following theorem immediately follows from expression (3.8)

Theorem 4.1. *If the set (3.1) is convex, then for optimality of the admissible control $u(t,x)$ the inequality*

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1} \Delta_{v(t,x)} H(t,x,z(t,x),u(t,x),\psi(t,x)) \le 0 \tag{4.1}$$

should be fulfilled for all $v(t,x) \in U$, $(t,x) \in T \times X$.

Theorem 4.1 is an analogue of Pontryagins discrete maximum principle [22-25] for the considered problem and is a first order necessary optimality condition. Therefore, the number of non-optimal controls satisfying the maximum condition (4.1) may be sufficiently great. Besides, possibility of degeneration of optimality condition (4.1) (see [26]) is not excluded.

Now we investigate the case of degeneration of necessary optimality condition (4.1).

Definition 4.1. The admissible control $u(t,x)$ is called singular control in the sense of Pontryagins maximum principle, if the relation

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1} \Delta_{v(t,x)} H(t,x,z(t,x),u(t,x),\psi(t,x)) = 0 \tag{4.2}$$

is fulfilled for all $v(t,x) \in U$, $(t,x) \in T \times X$. By definition, the singular controls satisfy first order necessary optimality conditions and consequently to analyze them from the optimality point of view we need second order and sometimes higher order optimality conditions [26].

Allowing for (4.2), the following statement follows from expression (3.8).

Theorem 4.2. *If the set (3.1) is convex, then for optimality of the singular control $u(t,x)$ the inequality*

$$y'(t_1,x_1)\varphi_{zz}(z(t_1,x_1))y(t_1,x_1) - \sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[y'(t,x)H_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$+2\Delta_{v(t,x)}H'_z(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)\Big] \ge 0 \tag{4.3}$$

should be fulfilled for all $v(t,x) \in U$, $(t,x) \in T \times X$.

Here, $y(t,x)$ is a solution of the equation in variations (3.4). Inequality (4.3) is a sufficiently general necessary optimality condition of singular controls. Based on this inequality, in some cases we can get constructively verifiable necessary optimality conditions of singular controls that are expressed obviously by the parameters of the problem (2.1)-(2.3).

The equations in variations (3.4) is a Volterra type linear, nonhomogeneous two-dimensional difference equation.

Using the scheme of the papers [1, 3-7], it is proved that the solution of the equation in variations (3.4) $y(t,x)$ allows the representation

$$y(t,x)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Big[\Delta_{v(\tau,s)}f_z(t,x,\tau,s,z(\tau,s),u(\tau,s))$$
$$+\sum_{\alpha=\tau}^{t}\sum_{\beta=s}^{x}R(t,x,\alpha,\beta)\Delta_{v(\tau,s)}f(\alpha,\beta,\tau,s,z(\tau,s),u(\tau,s))\Big]. \qquad (4.4)$$

Here, $R(t,x,\tau,s)$ is a solution of the Volterra type linear nonhomogeneous matrix difference equation

$$R(m,\ell,t,x)=\sum_{\tau=t}^{m}\sum_{s=x}^{\ell}R(m,\ell,\tau,s)f_z(\tau,s,t,x,z(t,x),u(t,x))$$
$$-f_z(m,\ell,t,x,z(t,x),u(t,x)). \qquad (4.5)$$

Equation (4.5) is a discrete analogue of the resolvent of Volterra type integral equation. By means of the scheme, for example of the paper [1], it is proved that $R(m,\ell,t,x)$ is also a solution of the equation

$$R(m,\ell,t,x)=\sum_{\tau=t}^{m}\sum_{s=x}^{\ell}f_z(m,\ell,\tau,s,z(\tau,s),u(\tau,s))R(\tau,s,t,x)$$
$$-f_z(m,\ell,t,x,z(t,x),u(t,x)). \qquad (4.6)$$

By analogy with the papers [1, 3-7], we call the matrix function $R(m,\ell,t,x)$ a resolvent of the equation in variations (3.4) and equations (4.5), (4.6) the equations of the resolvent. Assume that the right-hand side of system (2.3) has the form:

$$f(t,x,\tau,s,z,u)=A(t,x,\tau,s)\,g(\tau,s,z,u). \qquad (4.7)$$

Then representation (4.4) takes the form

$$y(t,x)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Big[A(t,x,\tau,s)\,\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s))$$
$$+\sum_{\alpha=\tau}^{t}\sum_{\beta=s}^{x}R(t,x,\alpha,\beta)A(\alpha,\beta,\tau,s)\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s))\Big]$$
$$=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\left\{\left[A(t,x,\tau,s)+\sum_{\alpha=\tau}^{t}\sum_{\beta=s}^{x}R(t,x,\alpha,\beta)A(\alpha,\beta,\tau,s)\right]\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s))\right\}.$$

Assuming

$$Q(t,x,\tau,s)=A(t,x,\tau,s)+\sum_{\alpha=\tau}^{t}\sum_{\beta=s}^{x}R(t,x,\alpha,\beta)A(\alpha,\beta,\tau,s),$$

this formula can be written in the form

$$y(t,x)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}Q(t,x,\tau,s)\,\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s)). \tag{4.8}$$

It is clear that from representation (4.8) we have

$$y(t_1,x_1)=\sum_{\tau=t_0}^{t_1}\sum_{\alpha=t_0}^{t_1}Q(t_1,x_1,\tau,s)\,\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s)).$$

Therefore we get

$$y^{'}(t_1,x_1)\varphi_{zz}(z(t_1,x_1))y(t_1,x_1)=\sum_{\tau=t_0}^{t_1}\sum_{s=x_0}^{x_1}\sum_{\alpha=t_0}^{t_1}\sum_{\beta=x_0}^{x_1}\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s))^{'}$$
$$\times\varphi_{zz}(z(t_1,x_1))\Delta_{v(\alpha,\beta)}g(\alpha,\beta,z(\alpha,\beta),u(\alpha,\beta)). \tag{4.9}$$

Thus,

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Delta_{v(t,x)}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$=\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Delta_{v(t,x)}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))Q(t,x,\tau,s)$$
$$\times\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s))\Big]. \tag{4.10}$$

Finally, using the scheme of the papers [20, 21], we prove the identity

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}y^{'}(t,x)H_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$=\sum_{\tau=t_0}^{t_1}\sum_{s=x_0}^{x_1}\sum_{\alpha=t_0}^{t_1}\sum_{\beta=x_0}^{x_1}\Delta_{v(\tau,s)}g^{'}(\tau,s,z(\tau,s),u(\tau,s))$$
$$\times\left\{\sum_{t=max(\tau,\alpha)}^{t_1}\ \sum_{x=max(s,\beta)}^{x_1}Q(^{'}t,x,\tau,s)\,H_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))Q(t,x,\alpha,\beta)\right\}$$
$$\times\Delta_{v(\alpha,\beta)}g(\alpha,\beta,z(\alpha,\beta),u(\alpha,\beta)). \tag{4.11}$$

Taking into account identities (4.9)-(4.11) in inequality (4.3), we get the relation

$$\sum_{\tau=t_0}^{t_1}\sum_{s=x_0}^{x_1}\sum_{\alpha=t_0}^{t_1}\sum_{\beta=x_0}^{x_1}\Delta_{v(\tau,s)}g^{'}(\tau,s,z(\tau,s),u(\tau,s))M(\tau,s,\alpha,\beta)\Delta_{v(\alpha,\beta)}g(\alpha,\beta,z(\alpha,\beta),u(\alpha,\beta))$$
$$+2\sum_{\tau=t_0}^{t_1}\sum_{s=x_0}^{x_1}\left[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Delta_{v(t,x)}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))Q(t,x,\tau,s)\right.$$
$$\left.\times\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s))\right]\leq 0 \tag{4.12}$$

where

$$M(\tau, s, \alpha, \beta) = -Q^{'}(t_1, x_1, \tau, s)\, \varphi_{zz}(z(t_1, x_1))\, Q(t_1, x_1, \alpha, \beta)$$

$$+ \sum_{t=max(\tau,\alpha)}^{t_1} \sum_{x=max(s,\beta)}^{x_1} Q(^{'} t, x, \tau, s)\, H_{zz}(t, x, z(t,x), u(t,x), \psi(t,x)) Q(t, x, \alpha, \beta). \quad (4.13)$$

Now we formulate the obtained result.

Theorem 4.3. *If the function $f(t, x, \tau, s, z, u)$ has the form (4.7) and the set (3.1) is convex, then for optimality of the singular control $u(t,x)$ the inequality (4.12) should be fulfilled for all $v(t,x) \in U$, $(t,x) \in T \times X$.*

R E F E R E N C E S

1. Kolmanovskiy V.B. On asymptotic equivalence of solutions of some Volterra difference equations. (Russian) *Avtomatica i Telemexanika*, **4** (2001), 47-56.

2. Lakshmikantham V., Trigiante D. Theory of difference equations. *Academic Press*, 1988.

3. Kolmanovskiy V.B. On limit periodicity of solutions for some Volterra systems. (Russian) *Avtomatika i Telemexanika*, **5** (2001), 36-43.

4. Ivinskaya E.V., Kolmanovskiy V.B. On boundedness of solutions for some Volterra difference equations. (Russian) *Avtomatika i Telemexanika*, **8** (2000), 86-97.

5. Song Y., Baker C.T.H. Perturbation theory for discrete Volterra equations. *Journ. Difference Equ. Appl.*, **9**, 10 (2003), 969-987.

6. Choi S.K., Goo Y.H., Koo N.J. Asymptotik behavior of nonlinear Volterra difference systems. *Bull. Korean Math. Soc.*, **44**, 1 (2007), 177-184.

7. Zouyousefain M., Leela S. Stability results for difference equations of Volterra type. *Applid. Math. Comput.*, **36** (1990), 51-61.

8. Butkovskiy A.Q. Theory of optimal control of the systems with distributed parameters. (Russian) *Nauka, Moscow*, 1965.

9. Vinokurov V.R. Optimal control of systems described by integral equations. (Russian) *Izv. Vuzov. Ser. mat.*, **4** (1967), 21-33.

10. Carlson D.A. An elementary proof of the maximum principle for optimal control problems governed by a Volterra inteqral equations. *Journal Optimiz. Theory and Appl.*, **54**, 1 (1987), 43-61.

11. Abdullayev A.A., Mansimov K.B. Second order necessary optimality conditions for the processes described by a system of Volterra type nonlinear integral equations. (Russian) *Avtomatika i Telemexanika*, **1** (2000), 3-11.

12. Choi K.S., Goo Y.H., Koo N.J. Existence and boundedness of solutions for Volterra discrete equations. *Journal of the Chungcheong Mathematical Society*, **19**, 3 (2006), 237-244.

13. Elaydi S.N., Kocic V.L. Global stability of nonlinear Volterra difference system. *Differential Equations Dynam. Systems*, **2** (1994), 87-90.

14. Baker C.T.H., Song Y. Discrete Volterra equations Discrete Volterra Operators, Fixed Point Theorems and their Application. *Numerical Analysis Report*, **398**, 22 (2002).

15. Qayshun I.V., Goryachkin V.V. Solvability and controllability conditions of linear two-parametric discrete systems. (Russian) *Differen. Uravn.*, **24**, 12 (1988), 2047-2051.

16. Qayshun I. V. Differential-difference two-parametric control systems. (Russian) *Differen. Uravn.*, **30**, 6 (1994), 821-938.

17. Dymkov M.P., Qayshun I.V. Optimal control problem of two-parametric system in Hilbert spare. (Russian) *Vychisl. i Prikl. Mat.*, **80** (1996), 10-23.

18. Vasilev O.V., Kirillova F.M. On optimal processes in two-parametric discrete systems. (Russian) *Dokl. AN SSSR.*, **173**, 1 (1967), 17-19.

19. Dymkov M.P. Extremal problems in multi parametric control systems. (Russian) *Belarus State University, Minsk*, 2005.

20. Mansimov K.B. Optimization of a class of discrete two-parametric systems. (Russian) *Diferen. Uravn.*, **27**, 2 (1991), 359-361.

21. Mansimov K.B. Necessary optimality conditions of second order in discrete two-parametric systems. (Russian) *Izv. AN Azerb. Ser. Fiz.-Tech. i Mat. Nauk.*, **2** (1998), 56-60.

22. Mansimov K.B. Discrete systems. (Russian) *Baku State University, Baku*, 2002.

23. Gabasov R., Kirillova F.M. Optimization methods. (Russian) *Belarus State University, Minsk*,

24. Propoy A.I. Elements of theory of optimal discrete processes. (Russian) *Nauka, Moscow*, 1973.

25. Ashepkov L.T. Optimal control of discontinuous systems. (Russian) *Nauka, Moscow*, 1987.

26. Gabasov R., Kirillova F.M. Singular optimal controls. (Russian) *Nauka, Moscow*, 1973.

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