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ON SOME SUBBASES OF A STRONG DIFFERENTIAL BASIS

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Abstract. The equivalence of a strong differential basis and some of its subbases is studied.

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Let $x \in \mathbb{R}^n$ and $\mathcal{B}(x)$ be a differential basis at the point x, i.e., a family containing the point x of sets of positive measure such that there exists a sequence (B_k) $(B_k \in \mathcal{B}(x))$ for which $\lim\limits_{k\to\infty} \operatorname{diam} B_k = 0$.

The family $\mathcal{B} = \bigcup\limits_{x\in\mathbb{R}^n} \mathcal{B}(x)$ is called a differential basis in $\mathbb{R}^n$ (see [1], p. 64).

For a function $f \in L_{loc}(\mathbb{R}^n)$ we denote by $\overline{D}_{\mathcal{B}}(\int f, x)$ and $\underline{D}_{\mathcal{B}}(\int f, x)$ the upper and the lower derivative, respectively, of the number at the point x of the integral of a function f with respect to the basis $\mathcal{B}$, i.e.,

$$\overline{D}_{\mathcal{B}}(\textstyle\int f, x) = \overline{\lim\limits_{B\to x}} \dfrac{1}{|B|_n} \displaystyle\int_B f = \lim_{r\to 0+} \sup_{\substack{B\in\mathcal{B}(x)\\ \operatorname{diam} B<r}} \frac{1}{|B|_n} \int_B f,$$

$$\underline{D}_{\mathcal{B}}(\textstyle\int f, x) = \underline{\lim\limits_{B\to x}} \dfrac{1}{|B|_n} \displaystyle\int_B f = \lim_{r\to 0+} \inf_{\substack{B\in\mathcal{B}(x)\\ \operatorname{diam} B<r}} \frac{1}{|B|_n} \int_B f.$$

If $\overline{D}_{\mathcal{B}}(\int f, x) = \underline{D}_{\mathcal{B}}(\int f, x) = f(x)$, then $\int f$ is called differentiable at the point x with respect to the basis $\mathcal{B}$.

Let $\mathcal{F}$ be some set of locally integrable functions. The differential bases $\mathcal{B}$ and $\mathcal{B}'$ are called equivalent on the set $\mathcal{F}$ if for each function $f \in \mathcal{F}$ the equalities

$$\overline{D}_{\mathcal{B}}(\textstyle\int f, x) = \overline{D}_{\mathcal{B}'}(\int f, x) \quad \text{and} \quad \underline{D}_{\mathcal{B}}(\int f, x) = \underline{D}_{\mathcal{B}'}(\int f, x)$$

are fulfilled almost everywhere.

When the differential bases $\mathcal{B}$ and $\mathcal{B}'$ are equivalent on the set $\mathcal{F}$, we will write $\mathcal{B} \overset{\mathcal{F}}{\sim} \mathcal{B}'$, and $\mathcal{B} \overset{\mathcal{F}}{\nsim} \mathcal{B}'$ otherwise ([2]).

Let us consider the following differential bases:

$\mathcal{K}$ is a differential basis for which $\mathcal{K}(x)$ consists of all cubic intervals containing the point x.

$\mathcal{J}$ is a strong differential basis, i.e., a basis for which $\mathcal{J}(x)$ consists of all intervals of the form $\prod\limits_{j=1}^{n} [a_i, b_i]$ $(a_i < b_i,\ i = \overline{1, n})$ containing the point x.

$\mathcal{J}_*$ is a strong symmetrical basis, i.e., a basis for which $\mathcal{J}_*(x)$ consists of all intervals of the basis $\mathcal{J}$ the center of which is the point x.

$\mathcal{J}_i$ $(i = \overline{1, 2^n})$ is a strong one-sided differential basis, i.e., a basis for which $\mathcal{J}_i(x)$ consists of all intervals of the basis $\mathcal{J}$ the i-th vertex of which is at the point x.

θ is a binary basis, i.e., a basis for which $\theta(x)$ consists of all intervals of the basis $\mathcal{J}$ the rib lengths of which are numbers of the form 2^m, $m \in \mathbb{Z}$.

θ_* is a binary symmetrical basis.

θ_0 is a binary-rational basis, i.e., a basis consisting of intervals of the form

$$\prod_{i=1}^{n} \left((k_i - 1)2^{m_i}; k_i 2^{m_i}\right), \quad k_i, m_i \in \mathbb{Z}, \quad i = \overline{1, n}$$

(constancy intervals of the rectangular partial sums of a Fourier-Haar series).

From the fundamental Lebesgue theorem [3] and the well-known results of Saks [4] and Busemann-Feller [5] it immediately follows that $\mathcal{K} \overset{L^+}{\not\sim} \mathcal{J}$, where L^+ is the set of non-negative locally integrable functions.

When studying the problems of convergence of multiple Fourier-Haar series, in [6] we in fact proved that

$$\theta_0 \overset{L^+}{\sim} \mathcal{J} \quad \text{and} \quad \theta_0 \overset{L}{\not\sim} \mathcal{J},$$

where L is the set of locally integrable functions

In [7] we also established that $\mathcal{J}_* \overset{L}{\sim} \mathcal{J}$ (as different from binary bases for which $\theta_* \overset{L}{\not\sim} \theta$) and $\mathcal{J}_i \overset{L}{\not\sim} \mathcal{J}$.

The question of equivalence remained open in the case where "the support point" lies at a point differing both from the center of the corresponding interval and from its vertices.

In the paper we will answer this question.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in [0,1]^n$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $n \geq 2$. Denote by the set of all intervals of the form $\prod_{i=1}^{n} [a_i, b_i]$ for which $a_i + \alpha_i(b_i - a_i) = x_i$, $i = \overline{1, n}$, and consider the differential basis $\mathcal{J}_\alpha = \bigcup_{x \in \mathbb{R}^n} \mathcal{J}_\alpha(x)$.

Note that $\mathcal{J}_{(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})} = \mathcal{J}_*$, while for $\alpha_i(1 - \alpha_i) = 0$ for all $i \in \{1, 2, \ldots, n\}$ the basis $\mathcal{J}_\alpha$ coincides with one of the one-sided bases.

We have established the validity of a more general theorem than the previously proved theorem on the equivalence of the bases $\mathcal{J}_*$ and $\mathcal{J}$.

Theorem 1. *If* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in [0,1]^n$, $n \geq 2$ *and* $\alpha_i(1 - \alpha_i) > 0$ *for all* $i \in \{1, 2, \ldots, n\}$, *then* $\mathcal{J}_\alpha \overset{L}{\sim} \mathcal{J}$.

Thus if "the support point" is the internal point of the corresponding intervals, then the basis $\mathcal{J}_\alpha$ is equivalent to the basis $\mathcal{J}$.

As has already been said, the equivalence of the bases does not take place when "the support point" lies at the vertex of the corresponding interval.

In the case where "the support point" lies on the boundary of the corresponding interval, but is not the vertex of this interval, the situation is somewhat different. In this direction we have succeeded in establishing the validity of the following theorem.

Theorem 2. *Let a number* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in [0,1]^n$, $(n \geq 2)$ *be such that there exists* $j \in \{1, 2, \ldots, n\}$ *for which* $\alpha_j(1 - \alpha_j) > 0$, *and* $\alpha_i(1 - \alpha_i) = 0$ *for all* $i \neq j$. *Then*

1) $\mathcal{J}_\alpha \overset{L}{\sim} \mathcal{J}$ *for* $n = 2$;

2) $\mathcal{J}_\alpha \overset{L}{\not\sim} \mathcal{J}$ *for* $n > 2$.

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