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FUNDAMENTAL SOLUTION OF ELASTIC STEADY STATE OSCILLATION EQUATIONS

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Abstract. The system of differential equations of steady state oscillations of anisotropic elasticity are considered. By the generalized Fourier transform technique and with the help of the limiting absorbtion principle, we construct maximally decaying at infinity matrices of fundamental solutions explicitly. Their expressions contain surface integrals over a certain semi-sphere and a line integrals along the edge boundary of the semi-sphere. We investigate near field and far field properties of the fundamental matrices and show that they satisfy the generalized Sommerfeld-Kupradze type radiation conditions at infinity.

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The homogeneous system of differential equations of steady state oscillations of anisotropic elasticity reads as follows (see, e. g., [1])

$$\mathbb{C}(\partial,\omega)\,u := C(\partial)u + \omega^2 u = c_{kjpq}\partial_j\partial_q u_p + \omega^2 u = 0, \tag{1}$$

where $u = (u_1, u_2, u_3)^\top$ is the displacement vector (amplitude), $\omega > 0$ is the oscillation (frequency)parameter,

$$\begin{aligned} &\mathbb{C}(\partial,\omega) := C(\partial) + \omega^2 I_3 = [c_{kjpq}\partial_j\partial_q + \delta_{kp}\omega^2]_{3\times3},\\ &C(\partial) = [c_{kjpq}\partial_j\partial_q]_{3\times3}. \end{aligned} \tag{2}$$

Here $\partial_j = \frac{\partial}{\partial x_j}$, I_3 stands for the unit 3×3 matrix, δ_{kp} is the Kroneker delta, the superscript $\top$ denotes transposition, c_{kjpq} are elastic constants; $c_{kjpq} = c_{jkpq} = c_{pqkj}$, $k, j, p, q = 1, 2, 3$.

Let $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse generalized Fourier transform in the space of tempered distributions (Schwartz space $S'(\mathbb{R}^3)$) which for regular summable functions f and g reads as follows

$$\mathcal{F}_{x\to\xi}[f] = \int\limits_{\mathbb{R}^3} f(x)e^{ix\cdot\xi}dx, \quad \mathcal{F}^{-1}_{\xi\to x}[g] = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} g(\xi)e^{-ix\cdot\xi}d\xi, \tag{3}$$

where $x = (x_1, x_2, x_3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$. Note that for arbitrary Multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $f \in S'(\mathbb{R}^3)$

$$\mathcal{F}[\partial^\alpha f] = (-i\xi)^\alpha \mathcal{F}[f], \quad \mathcal{F}^{-1}[\xi^\alpha g] = (i\partial)^\alpha \mathcal{F}^{-1}[g].$$

Denote by $\Psi(x,\omega)$ the matrix of fundamental solutions of the operator $\mathbb{C}(\partial,\omega)$

$$\mathbb{C}(\partial,\omega)\Psi(x,\omega) = I_3\delta(x). \tag{4}$$

Here $\delta(\cdot)$ is the Dirac's delta distribution. By standard arguments we can show that

$$\begin{aligned}\Psi(x,\omega) &= \mathcal{F}^{-1}[\mathbb{C}^{-1}(-i\xi,\omega)] = \mathcal{F}^{-1}\Big[\frac{\mathbb{C}^{*}(-i\xi,\omega)}{\Phi(\xi,\omega)}\Big] \\ &= N(\partial_x,\omega)\mathcal{F}^{-1}\Big[\frac{1}{\Phi(\xi,\omega)}\Big] = N(\partial_x,\omega)\Gamma(x,\omega), \end{aligned} \qquad (5)$$

where $\mathbb{C}^{-1}(-i\xi,\omega)$ is the inverse to the symbol matrix $\mathbb{C}(-i\xi,\omega)$, $\mathbb{C}^{*}(-i\xi,\omega)$ is the corresponding matrix of cofactors, $\Phi(\xi,\omega) = \det\mathbb{C}(-i\xi,\omega)$, $N(\partial_x,\omega) = \Big[N_{kj}(\partial_x,\omega)\Big]_{3\times 3}$ is the formally adjoint matrix to the matrix $\mathbb{C}(\partial,\omega)$ i.e.,

$$N(\partial_x,\omega)\mathbb{C}(\partial,\omega) = \mathbb{C}(\partial,\omega)N(\partial_x,\omega) = \Phi(x,\omega)I_3.$$

It is clear that N_{kj} is the nonhomogeneous differential operator of order 4.
Assume that for any $\eta \in \Sigma_1$, where

$$\Sigma_1 = \Big\{\eta \in \mathbb{R}^3 \Big| |\eta| = 1\Big\},$$

the equation $\Phi(\xi,\omega) = 0$ (written in spherical coordinates) has three different roots $t_1,\ t_2,\ t_3$ with respect to $t = \frac{\rho^2}{\omega^2}$, $\rho = |\xi|$, so

$$\Phi(\xi,\omega) = -a(\eta)\prod_{j=1}^{3}(\rho^2 - \omega^2\mu_j^2), \qquad (6)$$

where $t_j = \mu_j^2(\eta),\ \ j = 1,2,3$ are the different roots of the equation $\Phi(\xi,\omega) = 0$ and

$$a(\eta) = \Big[\mu_1^2(\eta)\ \mu_2^2(\eta)\ \mu_3^2(\eta)\Big]^{-1},\ \ \eta \in \Sigma_1;\quad \mu_j(-\eta) = \mu_j(\eta),\ \ a(-\eta) = a(\eta).$$

Taking complex $\tau = \omega + i\varepsilon,\ \ \varepsilon \neq 0$ instead of $\omega > 0$, we can show that $\Phi(\xi,\tau) \neq 0$ and

$$\Gamma(x,\tau) = \mathcal{F}^{-1}[\Phi^{-1}(\xi,\tau)].$$

With the help of the Cauchy integral theorem for analytic function and the limiting absorbtion principle [2], we prove the following

Theorem 1. *The fundamental solution of (1) has the following form*

$$\Psi(x,\omega,1) = N(\partial_x,\omega)\sum_{q=1}^{3}\int_{\Sigma_x^+} F_q(\eta)e^{i(x\cdot\eta)\rho_q}\,d\Sigma_1, \qquad (7)$$

or

$$\Psi(x,\omega,2) = -N(\partial_x,\omega)\sum_{q=1}^{3}\int_{\Sigma_x^+} F_q(\eta)e^{-i(x\cdot\eta)\rho_q}\,d\Sigma_1, \qquad (8)$$

where

$$F_q(\eta) = -\frac{i}{8\pi^2}\frac{\rho_q(\eta)}{a(\eta)\Big\{\prod\limits_{j=1,\, j\neq q}^{3}[\rho_q^2(\eta)-\rho_j^2(\eta)]\Big\}}, \quad \eta\in\Sigma_1,$$

$\rho_q(\eta) = \omega\mu_q(\eta)$, $q = 1,2,3$ and $\Sigma_x^+ = \{\eta : \eta\in\Sigma_1 \text{ and } (x\cdot\eta)\geq 0\}$.

Clearly, $\Psi(x,\omega,2) = \overline{\Psi(x,\omega,1)}$.

Denote by S_q the characteristic surface given by the equation $\rho = \rho_q(\eta)$, $q = 1,2,3$, $\eta\in\Sigma_1$. We assume, that S_q is star-shaped surface with respect to the origin and convex; it means that $\xi\cdot\eta(\xi)\geq 0$ for all $\xi\in S_q$, where $n(\xi)$ is the outward unit normal vector at $\xi\in S_q$.

Note that $\eta\rho_q(\eta) = \xi\in S_q$ and

$$\rho_q^2\, d\Sigma_1 = \Big(\frac{\xi}{|\xi|}\cdot n(\xi)\Big)dS_q = \frac{1}{\rho_q}\Big(\xi\cdot n(\xi)\Big)dS_q,$$

so we can rewrite (7) in the equivalent form

$$\Psi(x,\omega,1) = N(\partial_x,\omega)\sum_{q=1}^{3}\int\limits_{S_q^+(x)}\frac{F_q(\eta)(\xi\cdot n(\xi))}{\rho_q^3(\eta)}\, e^{i\,(x\cdot\xi)}dS_q. \tag{9}$$

In this paper we essentially use the following

Lemma 1. *If*

$$\Phi(x) = \int\limits_{\Sigma_x^+}\varphi(x,\eta)d_\eta\Sigma_1$$

and $\varphi(\cdot,\eta)\in C^1(\mathbb{R}^3)$, *then*

$$\frac{\partial\Phi(x)}{\partial x_k} = \int\limits_{\Sigma_x^+}\frac{\partial\varphi(x,\eta)}{\partial x_k}\, d_\eta\Sigma_1 + \frac{1}{|x|}\int\limits_{\gamma_x}\varphi(x,\eta)\eta_k d\gamma \tag{10}$$

where $\gamma_x = \partial\Sigma_x^+$.

We prove the following

Theorem 2. *The fundamental solution* $\Psi(x,\omega,1)$ *of equation (1) is represented as*

$$\Psi(x,\omega,1) = \Psi^{(1)}(x) + \Psi^{(0)}(x), \tag{11}$$

where

$$\Psi^{(1)}(x) = \int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)e^{i(x\cdot\eta)\rho_q}\, d\Sigma_1, \tag{12}$$

$$\Psi^{(0)}(x) = -\frac{1}{8\pi^2|x|}\int\limits_{\gamma_x}C^{-1}(\eta)\, d\gamma_x; \tag{13}$$

here $C^{-1}(\eta)$ *is the inverse matrix of* $C(\eta)$. *Moreover, if* $|x| \to 0$, *then*

$$\frac{\partial}{\partial x_k}\Big[\Psi^{(1)}(x)\Big] = O(1); \quad \frac{\partial^2}{\partial x_k \partial x_j}\Big[\Psi^{(1)}(x)\Big] = O\Big(\frac{1}{|x|}\Big)$$

and

$$\lim_{|x|\to 0} \Psi^{(1)}(x) = \int\limits_{\Sigma_x^+} \sum_{q=1}^{3} F_q(\eta) N(i\eta\rho_q, \omega)\, d\Sigma_1.$$

Remark, that $\Psi^{(0)}(x)$ is the fundamental solution of the static equation ($\omega = 0$)

$$C(\partial)\Psi^{(0)}(x) = \delta(x) I_3.$$

Taking into account the results obtained in [2] and [3], we can show that for $|x| \gg 1$

$$\Psi(x,\omega,1) = \sum_{q=1}^{3}\Big\{ -\frac{1}{4\pi}\, \frac{N(i\xi^{(q)},\omega)}{a(\eta^{(q)}) \prod\limits_{j=1,\, j\neq q}^{3} [\rho_q^2(\eta^{(q)}) - \rho_j^2(\eta^{(q)})]} \times \frac{e^{i(x\cdot\xi^{(q)})}}{|\nabla_\eta \rho_q(\eta^{(q)})|\sqrt{k_q(\xi^{(q)})}} + O(|x^{-2}|)\Big\},$$

where $\eta^{(q)} = \frac{\xi^{(q)}}{|\xi^{(q)}|}$, $\xi^{(q)} \in S_q$, $n(\xi^{(q)}) = \frac{x}{|x|}$, and $\nabla_\eta = (\frac{\partial}{\partial\eta_1}, \frac{\partial}{\partial\eta_2}, \frac{\partial}{\partial\eta_{3,}})$.
Finally we prove the following

Theorem 3. *For* $|x| \gg 1$

$$\Psi(x,\omega,1) = \sum_{q=1}^{3} \overset{(q)}{\Psi}(x,\omega,1), \quad \overset{(q)}{\Psi}(x,\omega,1) = O(|x|^{-1}),$$

$$\frac{\partial \overset{(q)}{\Psi}(x,\omega,1)}{\partial x_k} - i\xi_k \overset{(q)}{\Psi}(x,\omega,1) = O(|x|^{-2}).$$

These conditions are called the generalized Sommerfeld-Kupradze type radiation conditions.

R E F E R E N C E S

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