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ON SOME CONNECTION OF GENERALIZED MÖBIUS-LISTING'S SURFACES GML_2^n WITH SETS OF KNOTS OR LINKS

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Abstract. We consider the cutting process of generalized Möbius-Listing's surfaces GML_2^n along a set of lines "parallel" to their "basic line". We show connections of the resulting mathematical objects with the set of knots and links.

Keywords and phrases: Möbius strip, Möbius-Listing's surfaces, knots, links.

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1. Notation and analytic representation of "regular" surfaces GML_2^n.

Without loss of generality and to simplify the proofs, in this article we consider the following notation and restrictions:

- X, Y, Z, or x, y, z is the ordinary notation for coordinates;
- τ, ψ, θ are local coordinates or parameters in parallelogram:

$$\begin{array}{ll} 1. & \tau \in [\tau_*, \tau^*], \text{ where } \tau_* \leq \tau^* \text{ usually are non-negative constants;} \\ 2. & \psi \in [0, 2\pi]; \\ 3. & \theta \in [0, 2\pi]. \end{array} \tag{1}$$

Generalized Möbius-Listing's surfaces GML_2^n are "regular" and have a circle as a basic line. This means that the parametric representations of these surfaces (6) or (6*) in [4] have the following simple form: [1]

$$\begin{aligned} X(\tau,\theta) &= \left[R + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\cos\theta, \\ Y(\tau,\theta) &= \left[R + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\sin\theta, \\ Z(\tau,\theta) &= \tau\sin\left(\psi + \frac{n\theta}{2}\right), \end{aligned} \tag{2}$$

where, respectively:

R is the constant radius of basic circle;

the "rule of twisting around basic line" is "regular";

n is the "number or twisting" of GML_2^n, it is an arbitrary integer number, i.e., the number defined by eq. (5) in [3,4] is $\mu \equiv n/2$;

in the present case, ψ is a constant defined in (1) (but when $n = 0$, the number ψ in eq. (3) defines even the type of the corresponding surface, for example: if $\psi = 0$,

[1] Note that in the present article n denotes the number of rotations and m the symmetry number of the cross section, while in [4] the meaning of these indices was reversed.

then the "regular" generalized Möbius-Listing's surfaces GML_2^0, with a circle as basic line, is a ring ($R > \tau^*$) or circle ($R = \tau^*$), and if $\psi = \dfrac{\pi}{2}$, then GML_2^0 is a cylinder, in other cases these surfaces are cones or truncated cones (see [2-4])).

• Everywhere in this article we use the term "Link-1" instead of "knot".

Definition 1. A closed line (similar to the basic or border's line) which is situated on a GML_2^n and is "parallel" to the basic (or border's) line of the GML_2^n, i.e., distance between this line and basic or border's lines is constant, is called a **"slit line"** or shortly an **"s-line"**.

• If the distance between an s-line and the basic line is zero, then this s-line coincides with the basic line (and sometimes is called **"B-line"**).

Definition 2. A domain situated on the surface GML_2^n and such that its border's lines are slit lines, is called a **"slit zone"** or shortly an **"s-zone"** .

• The distance between the border's lines of an s-zone is the **"width"** of this s-zone.

• If an s-zone's width equals to zero, then this zone reduces to an s-line.

Definition 3. If the **"B-line"** is properly contained inside a **"slit zone"**, i.e., his distance to the border's lines is strictly positive, then this **"slit zone"** will be called a **"B-zone"**.

Definition 4. The **"process of cutting"** or shortly the **"cutting"** is always realized along some s-lines and produces the vanishing (i.e., elimination) of the corresponding s-zone (which eventually reduces to an s-line).

• If a GML_2^n surface is cut along an s-line (sometimes $\longrightarrow^1$), then the corresponding vanishing zone will be called an **s-slit** .

• If a GML_2^n surface is cut along its B-line (sometimes $\longrightarrow^B$), then the corresponding vanishing zone will be called a **B-slit**.

• If the vanishing zone, after an **s-slit** (a **B-slit**) is given by an **"s-zone"** (a **"B-zone"**), then the cutting process will be called an **s-zone-slit** (a **B-zone-slit**).

• If a GML_2^n surface is cut $(\kappa+1)$-times along $(\kappa+1)$, $\kappa = 0, 1, 2, \dots,$ different s-lines and none of them coincides with the B-line (for this process we use the symbolic notation: $\longrightarrow^{\kappa+1}$), then the resulting object is called a "$(\kappa+1)$-slitting GML_2^n", and the corresponding vanishing zones are $(\kappa+1)$**-slits**. In this case the cutting process is called a $(\kappa+1)$**-zone-slits**.

• If a GML_2^n surface is cut $(\kappa+1)$-times along $(\kappa+1)$, $\kappa = 0, 1, 2, \dots,$ different s-lines and one of this line coincides with the B-line (for this process we use the symbolic notation: $\longrightarrow^{B+\kappa}$), then the resulting object is called a "$(B+\kappa)$-slitting GML_2^n", and the corresponding vanishing zones are $(B+\kappa)$**-slits**. In this case the cutting process is called a $(B+\kappa)$**-zone-slits**.

2. Relations between the set of generalized Möbius-Listing's surfaces and the sets of knots and links

Theorem 1. *If the GML_2^n surface is cut $(\kappa+1)$-times along $(\kappa+1)$ different (i.e. $\kappa = 0, 1, \dots$)s-lines, and n is an even integer, then for each κ, after $(B+\kappa)$-zone-slits or $(\kappa+1)$-zone-slits, an object "**Link-**$(\kappa+2)$" appears, whose each component is a GML_2^n*

surface (knot with structure $\{0_1\})^2$*, i.e., if* $n = 2\omega$*, then for each* $\omega = 0, 1, 2, \dots,$ *and* κ*:*

Case A.

$$GML_2^{2\omega} \longrightarrow^{B+\kappa} \ Link\text{-}(\kappa+2) \ \ of \ \ (\kappa+2) \times GML_2^{2\omega};$$

Case B.

$$GML_2^{2\omega} \longrightarrow^{\kappa+1} \ Link\text{-}(\kappa+2) \ \ of \ \ (\kappa+2) \times GML_2^{2\omega}.$$

Some examples are given in Figs. 1 and 2.

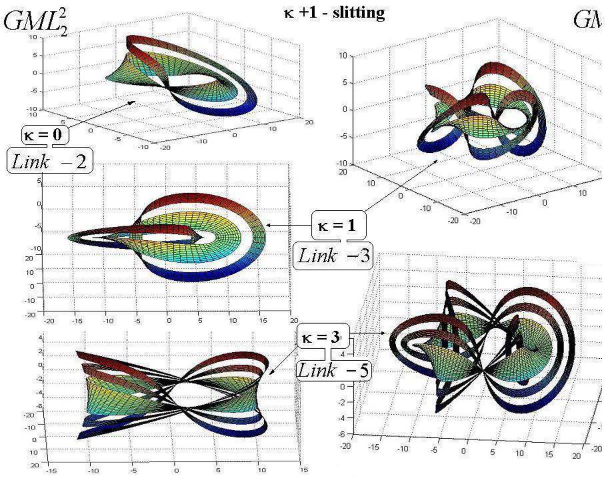

Fig.1.

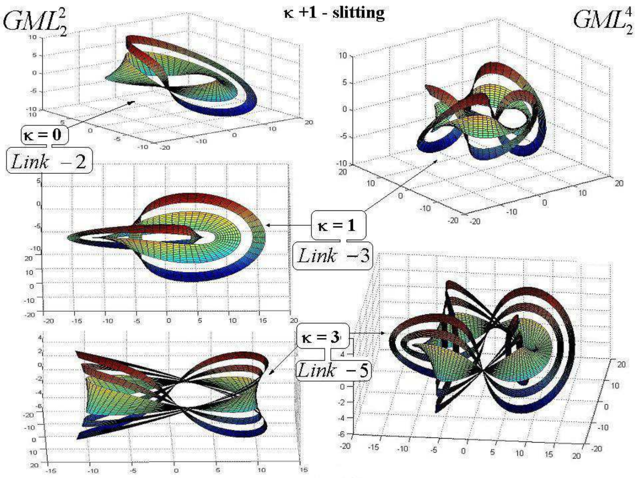

Fig.2

Theorem 2. *If the* GML_2^n *surface is cut* $(\kappa+1)$*-times along* $(\kappa+1)$ *different (i.e.* $\kappa = 0, 1, \dots$*)s-lines and* n *is an odd integer* $n = 2w+1$*, then for each* κ*, after*
Case A. $(B+\kappa)$*-zone-slits an object "**Link-**$(\kappa+1)$" appears, whose each component is a* GML_2^{2n+2} *surface (knot with structure* $\{n_1\}$*),*[3] *i.e., for each* $\omega = 0, 1, 2, \dots,$ *and* κ

$$GML_2^{2\omega+1} \longrightarrow^{B+\kappa} \ Link\text{-}(\kappa+1) \ \ of \ \ (\kappa+1) \times GML_2^{4\omega+4}.$$

Case B. $(\kappa+1)$*-zone-slits an object "**Link-**$(\kappa+2)$" appears, whose one component is a* GML_2^n *surface (knot with structure* $\{0_1\}$*), and each other component is a* GML_2^{2n+2} *surface (knot with structure* $\{n_1\}$*, except for* $n = 1$*, since in this case the topological group is* $(0)_1)$*,*[4]*; i.e., for each* $\omega = 0, 1, 2, \dots,$ *and* κ*,*

$$GML_2^{2\omega+1} \longrightarrow^{\kappa+1} \ Link\text{-}(\kappa+2) \ of \ one \ GML_2^{2\omega+1} \ and \ (\kappa+1) \times GML_2^{4\omega+4}.$$

Some examples are given in Figs. 3. and 4.

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[2]The topological group of the Link-$(k+2)$, in this case is at present unknown; only when $k = 0$, the link-2 is of type $\{n_1^2\}$, according to the standard classification (see [6-8]).

[3]The topological group of the Link-$(k+1)$ in this case is at present unknown; only when $k = 0$, the knot is of type $\{n_1\}$, when $n > 1$, and of type $\{0_1\}$, when $n = 1$, according to the standard classification (see [8]).

[4]The general topological group, in this case, is at present unknown.

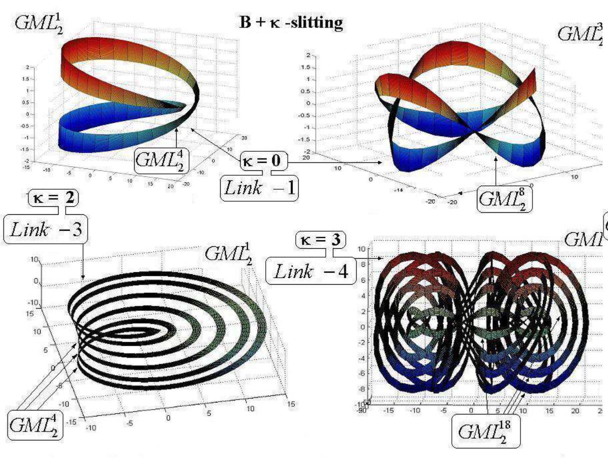

Fig.3.

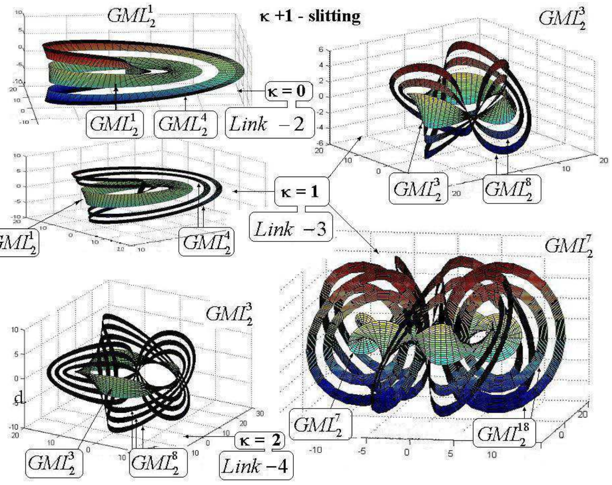

Fig.4

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