Seminar of I. Vekua Institute of Applied Mathematics
REPORTS, Vol. 35, 2009

EXISTENCE OF OPTIMAL INITIAL DATA AND WELL-POSEDNESS WITH RESPECT TO FUNCTIONAL FOR A CLASS OF DELAY OPTIMAL PROBLEM

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Abstract. Existence theorems of the optimal initial function and vector, the optimal initial moment and delays (optimal initial data) are obtained. The question of the continuity of the integral functional minimum (well-posedness with respect to functional) with respect to perturbations of the right-hand side of equation and integrand is investigated.

Keywords and phrases: Delay differential equations, existence theorems, well-posedness.

AMS subject classification (2000): 49J25; 34K35.

Let $0 < \tau_{1i} < \tau_{2i}, i = \overline{1,s}, a < t_1 < t_2 < t_3 < b$, be given numbers and $t_3 - t_2 > \tau = \max(\tau_{21}, ..., \tau_{2s})$, R^n be n-dimensional vector space of points

$$x = (x^1, ..., x^n)^T, |x|^2 = \sum_{i=1}^{n}(x^i)^2,$$

and $\Phi \subset R^n, X_0 \subset R^n$ be compact sets; let Δ be a set of measurable initial functions $\varphi(t) \in \Phi, t \in [a-\tau, t_2]$, and vector functions

$$F_i(t,x,y) = (f_i^0(t,x,y), f_i(t,x,y))^T \in R^{1+n}, i = \overline{1,s},$$

be continuous on the set $I \times R^n \times R^n$, where $I = [a,b]$, and continuously differentiable with respect to $x, y \in R^n$.

To each element

$$w = (\tau_1, ..., \tau_s, t_0, \varphi, x_0) \in W = [\tau_{11}, \tau_{21}] \times \cdots \times [\tau_{1s}, \tau_{2s}] \times [t_1, t_2] \times \Delta \times X_0$$

we assign the differential equation

$$\dot{x}(t) = \sum_{i=1}^{s} f_i(t, x(t), x(t-\tau_i)), t \in [t_0, t_1] \tag{1}$$

with the initial condition

$$x(t) = \varphi(t), t \in [t_0 - \tau, t_0), x(t_0) = x_0. \tag{2}$$

Definition 1. Let $w \in W$. A function $x(t) = x(t; w) \in R^n, t \in [t_0 - \tau, t_1]$ is called a solution corresponding to the element w, if it satisfies the condition (2), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies Eq.(1) almost everywhere on $[t_0, t_1]$.

By W_0 we denote the set of such elements $w \in W$ for which there exists the corresponding solution $x(t; w)$. In what follows we will assume that $W_0 \neq \emptyset$.

We note that, if the following condition

$$|f_x(t,x,y)| + |f_y(t,x,y)| \leq L, \forall (t,x,y) \in I \times R^n \times R^n$$

is fulfilled, where $L > 0$ is a given number,then $W_0 = W$.

Let consider the following functional

$$J(w) = \sum_{i=1}^{s} \int_{t_0}^{t_1} f_i^0(t, x(t), x(t-\tau_i))dt, w \in W_0, \quad (3)$$

where $x(t) = x(t; w)$.

Definition 2. An element $w_0 \in W_0$ is said to be optimal if

$$J(w_0) \leq J(w)$$

for any $w \in W_0$.

Theorem 1. *Let the following conditions hold:*

1) there exists a compact $K_0 \subset R^n$ such that

$$x(t; w) \in K_0, t \in [t_0, t_1] \ \ \forall w \in W_0;$$

2) for any $(\xi_i, x_i) \in I \times K_0, i = \overline{1, s}$, the set

$$\big\{(F_1(\xi_1, x_1, y), ..., F_s(\xi_s, x_s, y)) : y \in \Phi\big\} \subset R^{(1+n)s}$$

is convex. Then there exists an optimal element w_0 for the problem (1)-(3).

Theorem 2. *Let $f_i(t, x, y) = A_i(t, x)y$ and function $f_i^0(t, x, y)$ is convex with respect to y. Moreover, let the set Φ is convex and the condition 1) of Theorem 1 is fulfilled. Then there exists an optimal element w_0 for the problem (1)-(3).*

The Theorems 1,2 are proved by a scheme given in [1,2].

Let consider the perturbed optimal problem

$$\dot{x}(t) = \sum_{i=1}^{s} \big[f_i(t, x(t), x(t-\tau_i)) + g_{i\delta}(t, x(t), x(t-\tau_i))\big], t \in, [t_0, t_1],$$

$$x(t) = \varphi(t), t \in [t_0 - \tau, t_0), x(t_0) = x_0,$$

$$J(w; \delta) = \sum_{i=1}^{s} \int_{t_0}^{t_1} \big[f_i^0(t, x(t), x(t-\tau_i)) + f_{i\delta}^0(t, x(t), x(t-\tau_i))\big]dt \to \min,$$

where the functions $G_{i\delta}(t, x, y) = (g_{i\delta}^0(t, x, y), g_{i\delta}(t, x, y)), i = \overline{1, s}$, are continuous on the set $I \times R^n \times R^n$ and continuously differentiable with respect to $x, y \in R^n$.

Theorem 3. *Let the conditions of the Theorem 1 hold . Then for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that for an arbitrary functions $G_{i\delta}(t, x, y), i = \overline{1, s}$, satisfying the conditions:*

$$\sum_{i=1}^{s} \sup \Big\{ \int_{t_1}^{t_3} |G_{i\delta}(t, x, y)|dt : (x, y) \in K_1 \times K_1 \Big\} \leq \delta \quad (4)$$

and the set

$$\big\{(F_1(\xi_1, x_1, y) + G_{1\delta}(\xi_1, x_1, y), ..., F_s(\xi_s, x_s, y) + G_{s\delta}(\xi_s, x_s, y)) : y \in \Phi\big\}$$

is convex, where $K_1 \subset R^n$ *is a compact set containing some neighborhood of the set* $K_0 \cup \Phi$. *Then there exists an optimal element* $w_{0\delta}$ *for the perturbed optimal problem and the following inequality*

$$|J(w_{0\delta};\delta) - J(w_0)| \le \varepsilon \tag{5}$$

is fulfilled.

Theorem 4. *Let the conditions of Theorem 2 hold.For any* $\varepsilon > 0$ *there exists a number* $\delta > 0$ *such that for an arbitrary functions*

$$G_{i\delta}(t,x,y) = (g^0_{i\delta}(t,x,y), A_{i\delta}(t,x)y), i = \overline{1,s},$$

satisfying the condition (4) and the functions $g^0_{i\delta}(t,x,y), i = \overline{1,s}$, *are convex with respect to* y. *Then there exists an optimal element* $w_{0\delta}$ *for the perturbed optimal problem and the inequality (5) is fulfilled.*

At the end we note that the Theorems 3,4 are proved by a scheme given in [2].

Acknowledgement. The work was supported by the Georgian National Science Foundation, Grant No.GNSF/ST08/3-399.

R E F E R E N C E S

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Received 22.06.2009; revised 17.07.2009; accepted 22.09.2009.

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