Seminar of I. Vekua Institute of Applied Mathematics REPORTS, Vol. 35, 2009

ON THE ACCURACY OF THE METHOD OF SOLUTION OF A BOUNDARY VALUE PROBLEM FOR A TWO-DIMENSIONAL KIRCHHOFF EQUATION

Peradze J.

Abstract. The error of the method of solution of a nonlinear integro-differential equation describing the static state of a two-dimensional body is studied.

Keywords and phrases: Kirchhoff static equation, error of the method, estimation.

AMS subject classification (2000): 65L10; 65G; 65H05.

1. Statement of the Problem

Let us consider the following boundary value problem

$$\varphi\left(\int_\Omega \left(w_x^2+w_y^2\right)\,dx\,dy\right)(w_{xx}+w_{yy})=f(x,y), \tag{1}$$

$$(x,y)\in\Omega,$$

$$w(x,y)\big|_{\partial\Omega}=0, \tag{2}$$

where $\Omega=\{(x,y)\mid 0<x<\pi,\ 0<y<\pi\}$, $\partial\Omega$ is the boundary of the domain Ω, $\varphi=\varphi(z)$, $f=f(x,y)$ are the given functions and $w=w(x,y)$ is the function we want to define. It is assumed that $\varphi(z)$, $0\le z<\infty$, is a continuously differentiable function that satisfies the condition

$$\varphi(z)>\alpha>0,\quad 0\le z<\infty. \tag{3}$$

Equation (1) describes the static state of a two-dimensional body. When $\varphi(z)$ is a linear function, equation (1) is obtained by truncating the time argument t in a two-dimensional oscillation equation based on Kirchhoff's theory [1]. The introduction of the function $\varphi(z)$ makes it possible not to restrict the consideration to Hooke's law in the stress-strained relation [2].

2. Method of solution

To find $w(x,y)$ we will use M. Chipot's approach [3], [4]. The function $w(x,y)$ is written in the form

$$w(x,y)=\lambda v(x,y), \tag{4}$$

where λ and $v=v(x,y)$ are respectively the parameter and the function to be found. Substituting (4) into (1) we obtain

$$\lambda\varphi\left(\lambda^2\int_\Omega (v_x^2+v_y^2)\,dx\,dy\right)(v_{xx}+v_{yy})=f(x,y). \tag{5}$$

Without loss of generality, equation (5) is replaced by the system of equations

$$v_{xx} + v_{yy} = f(x,y),$$
$$\lambda\varphi\left(\lambda^2 \int_\Omega (v_x^2 + v_y^2)\, dx\, dy\right) = 1.$$

As seen from (2) and (4), the function $v(x,y)$ vanishes on the boundary. Therefore for this function we have the boundary value problem

$$v_{xx} + v_{yy} = f(x,y), \tag{6}$$
$$v\big|_{\partial\Omega} = 0, \tag{7}$$

while the parameter $\lambda > 0$ is defined as a solution of the equation

$$\lambda\varphi(s\lambda^2) = 1, \tag{8}$$

where

$$s = \int_\Omega (v_x^2 + v_y^2)\, dx\, dy > 0. \tag{9}$$

3. Errors of the method and their estimates

Suppose that the function $f(x,y)$ can be represented as a series

$$f(x,y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{ij} \sin ix \sin jy \tag{10}$$

for the coefficients of which the inequality

$$f_{ij}^2 \le \frac{\omega}{i^p j^q}\,, \quad i,j = 1,2,\dots\,, \tag{11}$$
$$\omega > 0, \quad p > 0,5,\ q > 0,5,$$

is fulfilled.

Let us expand the solution of problem (6), (7) into the series

$$v(x,y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} v_{ij} \sin ix \sin jy. \tag{12}$$

Using (6), (10) and (12) we have

$$v_{ij} = -\frac{f_{ij}}{i^2 + j^2}\,, \tag{13}$$
$$i = 1,2,\dots, \quad j = 1,2,\dots\,.$$

The substitution of (12) and (13) into (9) gives

$$s = \frac{\pi^2}{4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{f_{ij}^2}{i^2 + j^2}\,. \tag{14}$$

It is natural to assume that when using (12) and (14) we restrict our consideration to a finite number of summands. For the sake of simplicity we assume that in both formulas the operation of summation over i and 1 is carried out from 1 to n. As a result, instead of $v(x, y)$ and s we obtain their approximations

$$v_n(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} \sin ix \sin jy \tag{15}$$

and

$$s_n = \frac{\pi^2}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{f_{ij}^2}{i^2 + j^2}, \quad s_n \geq 0. \tag{16}$$

The solution of equation (8), where s is replaced by s_n, will not any longer be λ, but some value $\lambda_n > 0$.

Furthermore, since in general it is impossible to solve the nonlinear equation $\lambda\varphi(s_n\lambda^2) = 1$ exactly, we will have to use some iteration method giving an iteration approximation $\lambda_{n,k}$ where k is the number of an iteration step, $k = 0, 1, \ldots$. Therefore instead of the exact solution $w(x, y) = \lambda v(x, y)$ of problem (1), (2) obtained by virtue of (4) we have

$$w_{n,k}(x, y) = \lambda_{n,k} v_n(x, y). \tag{17}$$

Let us estimate the method error $w_{n,k}(x, y) - w(x, y)$ by using the values one part of which are obtained by the application of the method, while the other part can be calculated by means of the initial data of the problem.

Denote by $\| \cdot \|$ the norm in the space of functions $L_2(0, \pi; 0, \pi)$.

Since by (4) and (17) we have $\|w_{n,k}(x, y) - w(x, y)\| \leq |\lambda_{n,k} - \lambda| \, \|v_n(x, y)\| + |\lambda| \, \|v_n(x, y) - v(x, y)\|$ and, in addition to this, $|\lambda_{n,k} - \lambda| \leq |\lambda_{n,k} - \lambda_n| + |\lambda_n - \lambda|$, for the method error we obtain an estimate

$$\|w_{n,k}(x, y) - w(x, y)\| \leq (|\lambda_n - \lambda| + |\lambda_{n,k} - \lambda_n) \left(\|v_n(x, y)\| + \|\Delta v_n(x, y)\| \right) + |\lambda_{n,k}| \, \|\Delta v_n(x, y)\|, \tag{18}$$

where

$$\Delta v_n(x, y) = v_n(x, y) - v(x, y). \tag{19}$$

Let us consider formula (18). Its right-hand side contains the values $\|v_n(x, y)\|$ and $|\lambda_{n,k}|$ which can be calculated by the application of the method described here. As to the values $|\lambda_n - \lambda|$, $|\lambda_{n,k} - \lambda_n|$ and $\|\Delta v_n(x, y)\|$, we will discuss how to estimate them below.

a. estimation of $|\lambda_n - \lambda|$. Let us find out how the replacement of the parameter s by s_n affects the solution of equation (8). We rewrite this equation as

$$\lambda = \frac{1}{\varphi(s\lambda^2)} \tag{20}$$

and apply the Taylor formula. As a result we have

$$
\begin{gathered}
|\lambda_n-\lambda| \le \frac{M_1}{m_0^2}\,\max{}^2(\lambda_n,\lambda)|s_n-s|, \qquad (21)\\
m_0=\min_z|\varphi(z)|, \quad M_1=\max_z|\varphi'(z)|,\\
0\le z\le \max(s_n,s)\max{}^2(\lambda_n,\lambda).
\end{gathered}
$$

Let replace (21) by an expression that enables us to estimate $|\lambda_n-\lambda|$ by means of the values we have calculated. Using (20) and (3) we write the sought inequality in the form

$$
|\lambda_n-\lambda| \le \frac{M_1}{\alpha^2 m_0^2}\,\Delta s_n, \qquad (22)
$$

$$
m_0=\min_z|\varphi(z)|, \quad M_1=\max_z|\varphi'(z)|, \quad 0\le z\le \frac{1}{\alpha^2}\,(s_n+\Delta s_n).
$$

In (22) $\Delta s_n>0$ stands for the upper boundary of the error $|s_n-s|$ which is obtained by replacing formula (14) by the sum from (16). Let us calculate this value. Since

$$
|s_n-s|=\frac{\pi^2}{4}\bigg(\sum_{i=1}^{n}\sum_{j=n+1}^{\infty}\frac{f_{ij}^2}{i^2+j^2}+\sum_{i=n+1}^{\infty}\sum_{j=1}^{\infty}\frac{f_{ij}^2}{i^2+j^2}\bigg),
$$

by applying the Cauchy–Bunyakovski inequality, the integral test of series convergence and relations (10), (11) we have

$$
\Delta s_n \le \frac{\pi^2}{4}\big(\rho_1\Phi_1+\rho_2\Phi_2\big). \qquad (23)
$$

Here and further

$$
\begin{aligned}
\rho_{2l-1}&=\bigg(\int_n^\infty(1+x^2)^{-2l}\,dx+\int_n^\infty\int_1^n(x^2+y^2)^{-2l}\,dx\,dy\bigg)^{\frac12},\\
\rho_{2l}&=\bigg(\int_n^\infty(1+x^2)^{-2l}\,dx+\int_1^\infty\int_n^\infty(x^2+y^2)^{-2l}\,dx\,dy\bigg)^{\frac12}, \quad l=1,2\\
\Phi_1&=\omega\bigg(\int_n^\infty x^{-2q}\,dx+\int_n^\infty\int_1^n x^{-2p}y^{-2q}\,dx\,dy\bigg)^{\frac12},\\
\Phi_2&=\omega\bigg(\int_n^\infty x^{-2p}\,dx+\int_1^\infty\int_n^\infty x^{-2p}y^{-2q}\,dx\,dy\bigg)^{\frac12}.
\end{aligned} \qquad (24)
$$

Using the formulas for integrals of functions with respect to the argument z, which contain a^2+z^2 and $\operatorname{arctg}\dfrac{z}{a}$ (see [5]), we write

$$
\begin{aligned}
&\int\frac{1}{(1+x^2)^2}\,dx=\frac{x}{2(1+x^2)}+\frac12\operatorname{arctg}x, \quad \iint\frac{1}{(x^2+y^2)^2}\,dx\,dy\\
&=-\frac18\bigg[2\,\frac{1}{xy}+\bigg(\frac{1}{x^2}+3\,\frac{1}{y^2}\bigg)\operatorname{arctg}\frac{x}{y}+\bigg(3\,\frac{1}{x^2}+\frac{1}{y^2}\bigg)\operatorname{arctg}\frac{y}{x}\bigg].
\end{aligned} \qquad (25)
$$

By virtue of (24) and (25), in (23) we use the following relations

$$\begin{aligned}\rho_l &= \frac{1}{4}\bigg(7\pi + (2l-1)\,\frac{1}{n^2}\,\pi - 4\Big(\frac{1}{n} - (2-l)\,\frac{1}{n^2}\Big) - 8\,\frac{n}{1+n^2}\\ &\quad -2\Big(7+\frac{1}{n^2}\Big)\operatorname{arctg} n - 2\Big(1+3\,\frac{1}{n^2}\Big)\operatorname{arctg}\frac{1}{n}\bigg)^{\frac{1}{2}}, \quad l=1,2,\\ \Phi_1 &= \omega\bigg(\frac{1}{(2q-1)n^{2q-1}} + \frac{1}{(2p-1)(2q-1)}\Big(1-\frac{1}{n^{2p-1}}\Big)\frac{1}{n^{2q-1}}\bigg)^{\frac{1}{2}},\\ \Phi_2 &= \omega\bigg(\frac{1}{(2p-1)n^{2p-1}} + \frac{1}{(2p-1)(2q-1)}\,\frac{1}{n^{2p-1}}\bigg)^{\frac{1}{2}}.\end{aligned} \tag{26}$$

b. estimation of $|\lambda_{n,k}-\lambda_n|$. Let us consider one of the simple methods of constructing iteration approximations $\lambda_{n,k}$ and estimate the corresponding error [7]. Thus we solve the equation

$$\lambda\varphi(s_n\lambda^2) = 1 \tag{27}$$

whose exact solution is λ_n. Squaring equality (27) and multiplying by s_n, for $\mu = s_n\lambda^2$ we come to an equation of the form

$$g(\mu) = 0, \tag{28}$$

where $g(\mu) = \mu\varphi^2(\mu) - s_n$. As follows from (3), the function $g(\mu)$ changes its sign at the ends of the interval $\Big[0, \dfrac{s_n}{a^2}\Big]$, which implies that equation (28) has a solution on this interval. We apply the bisection method to (28) and construct a sequence of embedded into each other intervals $[a_0,b_0] \supset [a_1,b_1] \supset \cdots \supset [a_k,b_k] \supset \cdots$ such that $g(a_k)g(b_k) \le 0$, $b_k - a_k = \dfrac{1}{2^k}(b_0-a_0)$, $k=0,1,\ldots$, $a_0 = 0$, $b_0 = \dfrac{s_n}{a^2}$. The sequence (a_k), $k=0,1,\ldots$, will tend to a solution of (28). Taking into account the relationship between λ and μ, we conclude that $\lim\limits_{k\to\infty}\lambda_{n,k} = \lambda_n$ where $\lambda_{n,k} = \Big(\dfrac{a_k}{s_n}\Big)^{\frac{1}{2}}$. The rate of convergence is characterized by the relation

$$|\lambda_{n,k}-\lambda_n| \le \Big(\frac{a_k}{s_n} + \frac{1}{2^k\alpha^2}\Big)^{\frac{1}{2}} - \Big(\frac{a_k}{s_n}\Big)^{\frac{1}{2}}.$$

c. estimation of $\|\Delta v_n(x,y)\|$. By virtue of (19), (15), (12) and (13)

$$\|\Delta v_n(x,y)\| \le \frac{\pi}{2}\bigg[\bigg(\sum_{i=1}^{n}\sum_{j=n+1}^{\infty}\frac{f_{ij}^2}{(i^2+j^2)^2}\bigg)^{\frac{1}{2}} + \bigg(\sum_{i=n+1}^{\infty}\sum_{j=1}^{\infty}\frac{f_{ij}^2}{(i^2+j^2)^2}\bigg)^{\frac{1}{2}}\bigg].$$

As above, applying the Cauchy–Bunyakovski inequality, the integral test of series convergence and inequality (11), we obtain

$$\|\Delta v_n(x,y)\| \le \frac{\pi}{2}\big[(\rho_3\Phi_1)^{\frac{1}{2}} + (\rho_4\Phi_2)^{\frac{1}{2}}\big], \tag{29}$$

while for $l = 1, 2$ the definition of ρ_{2+l} is given in (24) and that of Φ_l and its value can be found in (24) and (26).

Again using some formulas from [5], we get the relations

$$\int \frac{1}{(1+x^2)^4}\,dx = \frac{x}{6(1+x^2)^3} + \frac{5x}{24(1+x^2)^2} + \frac{5x}{16(1+x^2)} + \frac{5}{16}\operatorname{arctg} x,$$

$$\iint \frac{1}{(x^2+y^2)^4}\,dx\,dy = -\frac{1}{16}\Big[-\frac{5}{18}\,\frac{1}{x^3y^3} + \frac{1}{2}\,\frac{1}{xy}\Big(\frac{1}{x^4}+\frac{1}{y^4}\Big)$$
$$+\frac{1}{3}\,\frac{1}{x^2+y^2}\Big(\frac{y}{x^5}+\frac{x}{y^5}\Big) + \frac{1}{3}\,\frac{1}{(x^2+y^2)^2}\Big(\frac{y}{x^3}+\frac{x}{y^3}\Big)$$
$$+\frac{5}{12}\Big(\frac{1}{x^6}+3\,\frac{1}{y^6}\Big)\operatorname{arctg}\frac{x}{y} + \frac{5}{12}\Big(3\,\frac{1}{x^6}+\frac{1}{y^6}\Big)\operatorname{arctg}\frac{y}{x}\Big]$$

enabling us to conclude that in (29)

$$\rho_l = \frac{1}{4\sqrt{3}}\bigg(\frac{75}{8}\pi - 8n\,\frac{1}{(1+n^2)^3} - \Big(11n+\frac{1}{n^3}\Big)\frac{1}{(1+n^2)^2} - \Big(16n+\frac{1}{n^5}\Big)\frac{1}{1+n^2}$$
$$+\Big((4-l)\,\frac{11}{3} + (2l-5)\,\frac{5}{8}\pi\Big)\frac{1}{n^6} + \frac{5}{6}\,\frac{1}{n^3} - \frac{3}{2}\Big(1+\frac{1}{n^4}\Big)\frac{1}{n}$$
$$-\frac{5}{4}\Big(15+\frac{1}{n^6}\Big)\operatorname{arctg} n - \frac{15}{4}\Big(\frac{1}{3}+\frac{1}{n^6}\Big)\operatorname{arctg}\frac{1}{n}\bigg)^{\frac{1}{2}}, \quad l = 3, 4.$$

The problem of the accuracy of a solution of a one-dimensional Kirchhoff static equation is considered in [6]-[8].

REFERENCES

1. Kirchhoff G. Vorlesungen über Mathematische Physik: Mechanik. *Teubner, Leipzig,* 1876.
2. Arosio A. Averaged evolution equations. The Kirchhoff string and its treatment in scales of Banach spaces. Functional analytic methods in complex analysis and applications to partial differential equations. (Trieste, 1993), 220–254, *World Sci. Publishing, River Edge, NJ,* 1995.
3. Chipot M. Elements of nonlinear analysis. Birkhäuser Advanced Texts: Basler Lehrbucher, [Birkhäuser Advanced Texts: Basel Textbooks] *Birkhäuser Verlag, Basel,* 2000.
4. Chipot M. Remarks on some class of nonlocal elliptic problems, *Recent advances of elliptic and parabolic issues. World Scientific,* 2006, 79-102.
5. Dwight H.B. Tables of integrals and other mathematical data. *4th ed. The Macmillan Company, New York,* 1961.
6. Abesadze T., Odisharia V., Peradze J. On the influence of a calculation error on the solution of the Kirchhoff nonlinear static equation. *Semin. I. Vekua Inst. Appl. Math. Reports*, **29** (2003), 3-10.
7. Peradze J., Papukashvili G. On one method of the silution of a non-linear integro-differential equation for a string. *Rep. Enl. Sess. Semin. I. Vekua Inst. Appl. Math.*, **22** 2008, 91-93.
8. Peradze J. On the errors of one method of solution of a static equation for a string, *Proc. A. Razmadze Math. Inst.* **148** (2008), 45-49.

Received 29.07.2009; revised 29.09.2009; accepted 26.10.2009.

Author's address:

J. Peradze
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: j_peradze@yahoo.com