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STUDY OF EXCENTRIC MULTI-LAYER LIQUID MOTION IN BIPOLAR-CYLINDRICAL COORDINATE SYSTEM

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Abstract. Excentric motion of viscose multi layer incompressible liquid in cylindric tubes with circle cross section is described. The thickness of flowing liquid layers differs along the circular coordinate. The solution of corresponding Navier-Stokes equations is constructed effectively and is expressed in a form of finite series for each layer.Obtained results are applied to mathematical simulation of blood flow through the narrow vessels.

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Introduction

Investigation of a viscous incompressible liquid flow in narrow cylindrical vessels [3] was initiated by Jean Louis Marie Poiseuille in his well-known works. However the excentric motion of such a liquid was not studied up to now. Moreover, in the present paper it is considered a multi-layer motion and at the same time it was assumed that the thickness of the flowing liquid layers varies along the circular coordinate, surrounding the axis of the vessel. In particular an attempt to construct a model of a blood flow distribution in a branching narrow vessel before liquid enters left and right subsidiary vessels was made. In this case the amount of erythrocytes in the ramus vessels is different and thus the symmetry of motion disturbs before branching [4]. Right this phenomena is described in the bipolar-cylindrical coordinate system.

Problem formulation

We are considering a steady multi-component motion of a viscous incompressible liquid in a bipolar-cylindrical system of coordinates ϱ, α, z ($h_\varrho = h_\alpha = \dfrac{c}{\cosh\varrho - \cos\alpha}$, $h_z = 1$ are Lame coefficients of the considered curvilinear orthogonal coordinate system) [2], assuming that the velocity vector $\overline{V}(u, v, w)$, where u, v, w are its projections on the normals to coordinate surfaces $\varrho = const, \alpha = const, z = const$, contains only a projection $w(\varrho, \alpha)$, that means that $u = v = 0$. In this case the condition of incompressibility fulfilled identically and from Navier-Stokes equations (gravity forces are absent) it follows

$$a)\ \frac{\partial P}{h\partial\varrho} = 0, \quad b)\ \frac{\partial P}{h\partial\alpha} = 0, \quad c)\ \frac{\partial P}{\partial z} = \mu\frac{1}{h^2}\left(\frac{\partial^2 w}{\partial\varrho^2} + \frac{\partial^2 w}{\partial\alpha^2}\right) \qquad (1)$$

$p = -1/(R_\varrho + A_\alpha + Z_z)$ is hydrostatic pressure, R_ϱ, A_α, Z_z are normal stresses; tangential stresses further will be denoted as $R_\alpha = A_\varrho, R_z = Z_\varrho, A_z = Z_\alpha$; μ is a coefficient of dynamic viscosity, individual for each layer. From $(1a)$ and $(1b)$ it follows, that $\dfrac{\partial p}{\partial z} = c_0 = const$. In this case

$$Z_\varrho = \frac{\mu(\cosh\varrho - \cos\alpha)}{c}\frac{\partial w}{\partial\varrho},\ R_\alpha = 0,\ R_\alpha = 0.$$

Let n-layer liquid flow along the parallel to Oz axe generatrix of cylindric surface, in the tube with cross-section $\Omega = \{\varrho_n < \rho < \infty,\ 0 < \alpha < \pi\}$ (here and further $\varrho > 0$). The cross-sections of layers are denoted as

$$\begin{aligned}\Omega_1 &= \{\varrho_1 < \varrho < \infty\}\\ \dots\\ \Omega_m &= \{\varrho_m < \varrho < \infty\}\\ \dots\\ \Omega_n &= \{\varrho_n < \varrho < \infty\}.\end{aligned}$$

Let us formulate the boundary-contact problem.

For the first layer we have

$$\Delta w_1 = \frac{p_0}{\mu_1},\quad \frac{\partial w_1}{\partial\alpha}\Big|_{\substack{\alpha=0\\ \alpha=\pi}} = 0,\quad w_1(\varrho_1,\alpha) = w_2(\varrho_1,\alpha),$$
$$\mu_1\frac{\partial w_1}{\partial\varrho}\Big|_{\varrho=\varrho_1} = \mu_2\frac{\partial w_2}{\partial\varrho}\Big|_{\varrho=\varrho_1},$$

For k-th layer, $k = 2,\dots,n-1$,

$$\Delta w_k = \frac{p_0}{\mu_k},\quad \frac{\partial w_k}{\partial\alpha}\Big|_{\substack{\alpha=0\\ \alpha=\pi}} = 0,\quad w_{k-1}(\varrho_{k-1},\alpha) = w_k(\varrho_{k-1},\alpha),$$
$$\mu_{k-1}\frac{\partial w_{k-1}}{\partial\varrho}\Big|_{\varrho=\varrho_{k-1}} = \mu_k\frac{\partial w_k}{\partial\varrho}\Big|_{\varrho=\varrho_{k-1}},$$

For n-th layer

$$\Delta w_n = \frac{p_0}{\mu_n},\quad \frac{\partial w_n}{\partial\alpha}\Big|_{\substack{\alpha=0\\ \alpha=\pi}} = 0,\quad w_{n-1}(\varrho_{n-1},\alpha) = w_n(\varrho_{n-1},\alpha),$$
$$\mu_{n-1}\frac{\partial w_{n-1}}{\partial\varrho}\Big|_{\varrho=\varrho_{n-1}} = \mu_n\frac{\partial w_n}{\partial\varrho}\Big|_{\varrho=\varrho_{n-1}}, w_n(\varrho_n,\alpha) = 0.$$

$$\text{where}\quad \Delta = \frac{(\cosh\varrho - \cos\alpha)^2}{c}\left(\frac{\partial^2 w}{\partial\varrho^2} + \frac{\partial^2 w}{\partial\alpha^2}\right).$$

From the conditions of equality of the normal stresses on the contact surfaces it follows, that for each m-th layer the function p is invariable and has a form $p = c_0 + p_0 z$, so a transition from one layer to another does not change constants c_0 and p_0.

Solution of boundary-contact problem

We will search for solution to the formulated boundary-contact problem of the form $w_m = w_m^* + w_m^0$ $(m = 1,2,\dots,n)$, where w_m^* is a particular solution of the equation

$\Delta w_m = \dfrac{p_0}{\mu_m}$ and w_m^0 is the general solution of the equation $\Delta w_m = 0$. Applying known expansion [1]

$$\ln 2(\cosh \varrho - \cos \alpha) = \varrho - 2 \sum_1^\infty \frac{1}{i} e^{-i\varrho} \cos(i\alpha)$$

we can obtain the representation for $(\cosh \varrho - \cos \alpha)^{-2}$ in the form of infinite series

$$\frac{1}{(\cosh \varrho - \cos \alpha)^2} = 2 \sum_i \frac{i + \cos \varrho}{\sinh^2 \varrho} e^{-i\varrho} \cos(i\alpha). \tag{2}$$

Applying (2) we finally will obtain the solution in a form

$$\begin{aligned} w_i &= a_{10} + \sum_i A_{1i} e^{-i\varrho} + \frac{c^2 p_0}{2\mu_1} \coth \varrho \left[1 + 2 \sum_i^\infty e^{-ip} \cos(i\alpha) \right], \\ w_m &= a_{m1} + a_{m2} + \sum_i \left[A_{mi} e^{i(\varrho_m - \varrho)} B_{mi} e^{i(\varrho - \varrho_m)} \right] \cos(i\alpha) \\ &+ \frac{c^2 p_0}{2\mu_m} \coth \varrho \left[1 + 2 \sum_i e^{-i\varrho} \cos(i\alpha) \right] \quad m = 2, 3, \ldots, n. \end{aligned} \tag{3}$$

It is clear, that obtained series converge exponentially in a closed area. If we bring into consideration a certain natural number i_0 providing the given accuracy of expansion (2), the summation of series introduced in the solution (3) could be produced from 1 to i_0. Constants $a_{10}, A_{1i} \ldots B_{mi}$ would be defined from the consistent system of linear algebraic equations of $(i_0 + 1)(2n - 1)$-th (or $2n(i_0 + 1)$-th) order. From the calculated values $Z_{\varrho m}$ and $Z_{\alpha m}$ could be found functions Z_ϱ and Z_α. On the base of obtained results, saying more precisely comparing them with the experimental data we hope to determine more accurately hemodynamic characteristics of blood microcirculation, in particular apparent viscosity, which appears to be the fundamental characteristic of blood flow resistance.

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