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NECESSARY CONDITIONS FOR EXISTENCE OF POSITIVE SOLUTIONS OF NONLINEAR DIFFERENCE EQUATIONS

Koplatadze R., Nanobashvili I.

Abstract. The difference equation

$$\Delta^2 u(k) + p(k)\left|u(\sigma(k))\right|^\lambda \operatorname{sign} u(\sigma(k)) = 0 \tag{0.1}$$

is considered, where $\lambda \in (0,1)$, $p : N \to R_+$, $\sigma : N \to N$, $\sigma(k) \geq k$ for $k \in N$ and difference operator is defined by $\Delta u(k) = u(k+1) - u(k)$, $\Delta^2 = \Delta \circ \Delta$.

Necessary conditions are obtained for the equation (0.1) to have a positive solution. Besides, oscillation criteria of new type are obtained.

Keywords and phrases: Oscillation, difference equations, advanced argument.

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1. Introduction

Consider the equation

$$\Delta^2 u(k) + p(k)\left|u(\sigma(k))\right|^\lambda \operatorname{sign} u(\sigma(k)) = 0, \tag{1.1}$$

where $p : N \to R_+$, $\sigma : N \to N$ are defined on the set $N = \{1, 2, \dots\}$ of the natural numbers, $\sigma(k) \geq k$ for $k \in N$, $\Delta u(k) = u(k+1) - u(k)$ and $\Delta^2 = \Delta \circ \Delta$.

Below it will assumed that

$$\sup\{p(i) : i \geq k\} > 0 \quad \text{for any} \quad k \in N.$$

For any $n \in N$, denote $N_n = \{n, n+1, \dots\}$.

Definition 1.1. Let $n_0 \in N$. A function $u : N_{n_0} \to R$ is said to be a proper solution of (1.1), if it satisfies (1.1) on N_{n_0} and

$$\sup\left\{|u(i)| : i \geq k\right\} > 0 \quad for \quad k \in N_{n_0}.$$

Definition 1.2. We say that a proper solution $u : N_{n_0} \to R$ of equation (1.1) is oscillatory, if for any $n \in N_{n_0}$, there are $n_1, n_2 \in N_n$ such that $u(n_1)\,u(n_2) \leq 0$. Otherwise, the proper solution is cold nonoscillatory.

In the present paper sufficient conditions for the oscillation of all proper solutions of (1.1) are established.

2. Main results

Below it is meant that the condition

$$\sum_{k=1}^{+\infty} k\,p(k) = +\infty \tag{2.1}$$

is fulfilled.

Let $k_0 \in N$. Denote by U_{k_0} the set of all proper solutions of (1.1) satisfying $u(k) > 0$ for $k \in N_{k_0}$.

Theorem 2.1. *Let $k_0 \in N$ and $U_{k_0} \neq \varnothing$. Then for any $\delta \in [0, \lambda]$ and for any $s \in N$*

$$\sum_{k=1}^{+\infty} k^{\lambda-\delta}\big(\rho_s(\sigma(k))\big)^{\delta} p(k) < +\infty,$$

where

$$\rho_1(k) = \sum_{\ell=1}^{k} \sum_{j=\ell}^{+\infty} p(j), \tag{2.2}$$

$$\rho_j(k) = \sum_{\ell=1}^{k} \sum_{j=\ell}^{+\infty} p(j)\big(\rho_{j-1}(\sigma(j))\big)^{\lambda}, \quad j = 2, \dots, s. \tag{2.3}$$

Theorem 2.1 play important role in establishing the sufficient conditions for the all proper solutions of equation (1.1) to be oscillatory.

Theorem 2.2. *Let condition* (2.1) *be fulfilled and for some $\delta \in [0, \lambda]$ and for some $s \in N$*

$$\sum_{k=1}^{+\infty} k^{\lambda-\delta}\big(\rho_s(\sigma(k))\big)^{\delta} p(k) = +\infty.$$

Then any proper solutions of equation (1.1) *is oscillatory, where ρ_s is defined by* (2.2) *and* (2.3).

Corollary 2.1. *Let*

$$\sum_{k=1}^{+\infty} k^{\lambda} p(k) = +\infty.$$

Then any proper solutions of equation (1.1) *is oscillatory.*

Corollary 2.2. *Let* (2.1) *is fulfilled and for some $s \in N$*

$$\sum_{k=1}^{+\infty} \big(\rho_s(\sigma(k))\big)^{\lambda} p(k) = +\infty.$$

Then any proper solutions of equation (1.1) *is oscillatory, where ρ_s is defined by* (2.2) *and* (2.3).

Theorem 2.3. *Let for some $\gamma \in (0, 1)$ and $\alpha \in (1, +\infty)$*

$$\liminf_{k \to +\infty} k^{\gamma} \sum_{i=k}^{+\infty} p(i) > 0$$

and

$$\liminf_{k \to +\infty} \frac{\sigma(k)}{k^{\alpha}} > 0.$$

Moreover, if at last one conditions

$$\alpha\,\lambda \geq 1$$

or

if $\alpha\,\lambda < 1$, *for some* $\varepsilon > 0$

$$\sum_{k=1}^{+\infty} k^{\frac{\alpha\,\lambda(1-\gamma)}{1-\alpha\,\lambda}-\varepsilon} p(k) = +\infty,$$

then any proper solutions of equation (1.1) *is oscillatory.*

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Authors' address:

R. Koplatadze and I. Nanobashvili
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: r_ koplatadze@yahoo.com