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ON A CONSTRACTION OF APPROXIMATE SOLUTIONS FOR THE GEOMETRICALLY AND PHYSICALLY NONLINEAR AND NON-SHALLOW SHELLS

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Abstract. In this paper the geometrically and physically nonlinear and non-shallow shells are considered. Under non-shallow shells will be meant 3-D shell-type elastic bodies satisfying the conditions $|hb_\alpha^\beta| \leq q < 1$ $(\alpha, \beta = 1, 2)$, in contrast to shallow shells, for which the assumption $hb_\alpha^\beta \cong 0$ is accepted, where h is the semi-thickness and b_α^β are mixed components of the curvature tensor of the shell's midsurface.

Using the method I. Vekua [1] and the method of a small parameter [2] 2-D system of equations for the nonlinear and non-shallow shells is obtained. For any approximation of order N the complex representation Vekua-Bitsadze [3] of the general solutions are obtained.

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A complete system of equilibrium equation and the stress-strain relations of the 3-D nonlinear theory of elasticity can be written as:

$$\nabla_i \boldsymbol{\sigma}^i + \boldsymbol{\Phi} = 0, \quad \boldsymbol{\sigma}^i = (E^{ijpq} + E^{ijpqsk} e_{sk}) e_{pq} (\mathbf{R}_j + \partial_j \mathbf{u}) \quad (i, ..., k = 1, 2, 3),$$

where ∇_i are covariant derivatives relatively to the space curvilinear coordinates x^i, $\boldsymbol{\sigma}^i$ and $\boldsymbol{\Phi}$ are, respectively, the contravariant "constituents" of the stress vector and an external force, e_{ij} are covariant components of the strain tensor, $\mathbf{u}$ is the displacement vector:

$$2e_{ij} = \mathbf{R}_i \partial_j \mathbf{u} + \mathbf{R}_j \partial_i \mathbf{u} + \partial_i \mathbf{u} \partial_j \mathbf{u}, \quad E^{ijpq} = \lambda g^{ij} g^{pq} + \mu (g^{ip} g^{jq} + g^{iq} g^{jp}),$$

$$E^{ijpqsk} = E_1 g^{ij} g^{pq} g^{sk} + E_2 (g^{ij} g^{pq} g^{sk} - g^{ij} g^{pk} g^{qs}) + E_3 g^{ip} g^{jq} g^{sk} + E_4 g^{is} g^{pq} g^{jk}.$$

Here $g^{ij} = \mathbf{R}^i \mathbf{R}^j$, λ and μ are Lame's constants and E_1, E_2, E_3, E_4 are modules of elasticity of the second order isotropic elastic bodies, $\mathbf{R}_i$ and $\mathbf{R}^i$ are covariant and contravariant basis vectors of the space domain Ω, which are connected with the basis vectors $\mathbf{r}_i$ and $\mathbf{r}^i$ of the midsurface S $(x^3 = 0)$ by the following relations:

$$\mathbf{R}_i = A_{i.}^{.j} \mathbf{r}_j, \quad \mathbf{R}^i = A_{.j}^{i.} \mathbf{r}^j, A_{\alpha.}^{.\beta} = a_\alpha^\beta - x_3 b_\alpha^\beta, \quad A_{i.}^{.3} = A_{.3}^{i.} = \delta_{i3},$$

$$A_{.\beta}^{\alpha.} = \vartheta^{-1} [a_\beta^\alpha + x_3 (b_\beta^\alpha - 2H a_\beta^\alpha)], \quad \vartheta = 1 - 2Hx_3 + Kx_3^2,$$

where $\mathbf{n} = \mathbf{r^3}$ is the normal of the midsurface S, H and K are middle and Gaussian curvatures of S, $a_\alpha^\beta = \mathbf{r}_\alpha \mathbf{r}^\beta$, $a_{\alpha\beta} = \mathbf{r}_\alpha \mathbf{r}_\beta$, $a = det\{a_{\alpha\beta}\}$, $(\alpha, \beta = 1, 2)$.

It should be noted that for the shallow shells we have:

$$\mathbf{R}_\alpha \cong \mathbf{r}_\alpha, \quad \mathbf{R}^\alpha \cong \mathbf{r}^\alpha, \quad g \cong a, \quad A_{\alpha.}^{.\beta} \cong a_\alpha^\beta, \quad A_{.\beta}^{\alpha.} \cong a_\beta^\alpha, \quad g = det\{\mathbf{R}_i \mathbf{R}_j\}.$$

For the Koiter-Naghdi refined theory of shells these relations have the form:

$$\mathbf{R}^{\alpha}=(a^{\alpha}_{\beta}+x_3 b^{\alpha}_{\beta})\mathbf{r}^{\beta}, \quad \mathbf{R}_{\alpha}=(a^{\beta}_{\alpha}-x_3 b^{\beta}_{\alpha})\mathbf{r}_{\beta}.$$

Now following I.Vekua we assume the validity of the expansions:

$$\Big(\vartheta\boldsymbol{\sigma}^i,\mathbf{u},\vartheta\mathbf{\Phi}\Big)=\sum_{m=0}^{\infty}\Big(h\overset{(m)}{\boldsymbol{\sigma}^i},h^2\overset{(m)}{\mathbf{u}},\overset{(m)}{\mathbf{\Phi}}\Big)P_m\Big(\frac{x_3}{h}\Big), \qquad (-h\le x_3\le h)$$

where P_m is Legendre polynomials of the order m.

To introduce a small parameter $\varepsilon=\frac{h}{R}$, where R is a certain radius of curvature of the midsurface S, will be obtained the following infinite system of 2-D equations:

$$h\nabla_{\alpha}\overset{(m)}{\sigma^{\alpha\beta}}-\varepsilon b^{\beta}_{\alpha}\overset{(m)}{\sigma^{\alpha 3}}R-(2m+1)\Big(\overset{(m-1)}{\sigma^{3\beta}}+\overset{(m-3)}{\sigma^{3\beta}}+\cdots\Big)+\overset{(m)}{F^{\beta}}=0,$$

$$h\nabla_{\alpha}\overset{(m)}{\sigma^{\alpha 3}}+\varepsilon b_{\alpha\beta}\overset{(m)}{\sigma^{\alpha\beta}}R-(2m+1)\Big(\overset{(m-1)}{\sigma^{3}_{3}}+\overset{(m-3)}{\sigma^{3}_{3}}+\cdots\Big)+\overset{(m)}{F^{3}}=0, \tag{1}$$

$$(\overset{(m)}{\sigma^{ij}}=\overset{(m)}{\boldsymbol{\sigma}^i}\mathbf{r}^j, \quad m=0,1,...)$$

where $\overset{(m)}{\mathbf{F}}=\overset{(m)}{\mathbf{\Phi}}+\frac{2m+1}{2h}\Big[\overset{(+)}{\vartheta}\overset{(+)}{\boldsymbol{\sigma}^3}-(-1)^m\overset{(-)}{\vartheta}\overset{(-)}{\boldsymbol{\sigma}^3}\Big],\overset{(\pm)}{\boldsymbol{\sigma}^3}=\boldsymbol{\sigma}^3(x^1,x^2,\pm h),\overset{(\pm)}{\vartheta}=1\mp 2Hh+Kh^2$.

The stress-strain relation have the form

$$\overset{(m)}{\boldsymbol{\sigma}^i}=\frac{1}{2}M^{i_1j_1p_1q_1}\Bigg\{\sum_{m_1=0}^{\infty}\Big[\overset{(m)}{A}{}^{ip}_{i_1p_1}\Big(hD_p\overset{(m_1)}{u_{q_1}}-\varepsilon RD_p\mathbf{r}_{q_1}\cdot\overset{(m_1)}{\mathbf{u}}\Big)+\overset{(m)}{A}{}^{iq}_{i_1q_1}\Big(hD_q\overset{(m_1)}{u_{p_1}} \tag{2}$$

$$-\varepsilon\, RD_q\mathbf{r}_{p_1}\cdot\overset{(m_1)}{\mathbf{u}}\Big)\Big]\mathbf{r}_{j_1}+\sum_{m_1,m_2=0}^{\infty}+\cdots\Bigg\}+\frac{1}{4}M^{i_1j_1p_1q_1s_1k_1}\Bigg\{\cdots+\sum_{m_1,\cdots m_5=0}^{\infty}\overset{(m)}{A}{}^{ijpqsk}_{i_1j_1p_1q_1s_1k_1}$$

$$\Big[\Big(hD_p\overset{(m_1)}{u_{p_2}}\cdot\mathbf{r}^{p_2}+\varepsilon RD_p\mathbf{r}^{p_2}\cdot\overset{(m_1)}{u_{p_2}}\Big)\cdots\Big(hD_j\overset{(m_5)}{u_{j_2}}\cdot\mathbf{r}^{j_2}+\varepsilon RD_j\mathbf{r}^{j_2}\cdot\overset{(m_5)}{u_{j_2}}\Big)\Big]\Bigg\},$$

where

$$D_i\overset{(m)}{\mathbf{u}}=\delta^{\beta}_i\nabla_{\beta}\overset{(m)}{\mathbf{u}}+\delta^3_i\overset{(m)}{\mathbf{u}'},\ \overset{(m)}{\mathbf{u}'}=(2m+1)\Big(\overset{(m+1)}{\mathbf{u}}+\overset{(m+3)}{\mathbf{u}}+\cdots\Big),\ \ D_{\alpha}\mathbf{r}_{\beta}=b_{\alpha\beta}\mathbf{n},$$

$$D_{\alpha}\mathbf{n}=-b^{\beta}_{\alpha}\mathbf{r}_{\beta},\ \ D_3\mathbf{r}_i=0,\ \ M^{i_1j_1p_1q_1}=\lambda a^{i_1j_1}a^{p_1q_1}+\mu\Big(a^{i_1p_1}a^{j_1q_1}+a^{i_1q_1}a^{j_1p_1}\Big),$$

$$\overset{(m)}{A}{}^{ip}_{i_1p_1}=\frac{2m+1}{2h}\int_{-h}^{h}\vartheta A^i_{i_1}A^p_{p_1}P_mP_ndx_3,\overset{(m)}{A}{}^{i\cdots k}_{i_1\cdots k_1}$$

$$=h^4\frac{2m+1}{2h}\int_{-h}^{h}\vartheta A^i_{i_1}\cdots A^k_{k_1}P_{m_1}\cdots P_mdx_3, \tag{3}$$

Integrals of the type (3) lend themselves to explicit calculation [4].

To find components of the displacement vector $\overset{(m)}{\mathbf{u}}$ and stress tensor $\overset{(m)}{\sigma}{}^{ij}$ we take of following series expansions with respect to the small parameter ε:

$$\Big(\overset{(m)}{\mathbf{u}},\overset{(m)}{\sigma}{}^{ij},\overset{(m)}{F}\Big)=\sum_{n=1}^{\infty}\Big(\overset{(m,n)}{\mathbf{u}},\overset{(m,n)}{\sigma}{}^{ij},\overset{(m,n)}{F}\Big)\varepsilon^n.\quad (m=0,1,\cdots,N)$$

Substituting the above expansions into the (1) and (2) than equalizing the coefficients of expansions for ε^n we obtain the following 2-D finite system of equilibrium equations with respect to components of displacement vector in the isometric coordinates $a_{11}=a_{22}=\Lambda(x^1,x^2)$, which has the form:

$$4\mu\partial_{\overline{z}}\Big(\Lambda^{-1}\partial_z\overset{(m,n)}{u}_+\Big)+2(\lambda+\mu)\partial_{\overline{z}}\overset{(m,n)}{\theta}+2\lambda\partial_{\overline{z}}\overset{(m,n)}{u'_3}-(2m+1)\mu$$

$$\times\Big[2\partial_{\overline{z}}\Big(\overset{(m-1,n)}{u_3}+\overset{(m-3,n)}{u_3}+\cdots\Big)+\overset{(m-1,n)}{u'_+}+\overset{(m-3,n)}{u'_+}+\cdots\Big]+\overset{(m,n)}{F_+}=0, \qquad (4)$$

$$\mu\Big(\nabla^2\overset{(m,n)}{u_3}+\overset{(m,n)}{\theta'}\Big)-(2m+1)\Big[\lambda\Big(\overset{(m-1,n)}{\theta}+\overset{(m-3,n)}{\theta}+\cdots\Big)$$

$$+(\lambda+2\mu)\Big(\overset{(m-1,n)}{u'_3}+\overset{(m-3,n)}{u'_3}+\cdots\Big)\Big]+\overset{(m,n)}{F_3}=0,$$

where $u_+=u_1+iu_2$, $\theta=\Lambda^{-1}\Big(\partial_z u_+\partial_{\overline{z}}\overline{u}_+\Big)$, $z=x^1+ix^2$,$2\partial_z=\partial_1-i\partial_2$, $\nabla^2=\frac{4}{\Lambda}\frac{\partial^2}{\partial z\partial\overline{z}}$.

Obviously, in passing from the n-th step of approximation to the $(n+1)$-th step only the right-hand of equations are changed. Below it will be omit upper index n. The general solution of the homogeneous system (4) we can find the form

$$\overset{(m)}{u}_+=\partial_{\overline{z}}\overset{(m)}{V}_++\Big(\frac{1}{\pi}\iint\limits_S\frac{\overline{\varphi_0'(\zeta)}-æ_1\varphi_0'(\zeta)dS_\zeta}{\overline{\zeta}-\overline{z}}-\overline{\psi_0'(z)}\Big)\delta_{0m}$$

$$-\Big(\frac{1}{\pi}\iint\limits_S\frac{\varphi_1'(\zeta)+\overline{\varphi_1'(\zeta)}dS_\zeta}{\overline{\zeta}-\overline{z}}+\eta_1\overline{\varphi_1''(z)}-2\overline{\psi_1'(z)}\Big)\delta_{1m}+æ_2\overline{\varphi_0''(z)}\delta_{2m}+\eta_2\overline{\varphi_1''(z)}\delta_{3m},$$

$$\overset{(m)}{u}_3=\overset{(m)}{V}_3-\Big(\frac{1}{\pi}\iint\limits_S(\varphi_1'(\zeta)+\overline{\varphi_1'(\zeta)})ln|\zeta-z|dS_\zeta-\psi_1(z)-\overline{\psi_1(z)}\Big)\delta_{0m} \qquad (5)$$

$$-\frac{3}{2}æ_2\Big[(\varphi_0'(z)+\overline{\varphi_0'(z)})\delta_{1m}-(\varphi_1'(z)+\overline{\varphi_1'(z)})\delta_{2m}\Big],\qquad (m=0,1,...,N)$$

$$\overset{(0)}{V}_1=\overset{(0)}{V}_2=0,\quad \overset{(0)}{u}_3=\psi_1(z)+\overline{\psi_1(z)},\quad if\ N=0,\quad (dS_\zeta=\Lambda(\zeta,\overline{\zeta})d\zeta d\overline{\zeta},\ \zeta=\xi+i\eta).$$

where $\varphi_0'(z)$,$\varphi_1'(z)$,$\psi_0'(z)$,$\psi_1'(z)$ are holomorphic functions of z and express the biharmonic solution of the system (4). Then $æ_1$, $æ_2$, η_1, η_2 are known constants.

Substituting expressions (5) into (4) the matrix equations for $\overset{(m)}{V_i}$ are obtained

$$\nabla^2 V - AV = X, \quad \nabla^2 \Omega - B\Omega = Y, \tag{6}$$

where V and Ω are column-matrices of the form

$$V = \big(\overset{(0)}{V_1}, \overset{(1)}{V_1}, ..., \overset{(N)}{V_1}, \overset{(0)}{V_3}, \overset{(1)}{V_3}, ..., \overset{(N)}{V_3}\big)^T, \qquad \Omega = \big(\overset{(0)}{V_2}, \overset{(1)}{V_2}, ..., \overset{(N)}{V_2}\big)^T,$$

and A and B are block-matrices $2N+2 \times 2N+2$ and $N \times N$ respectively.

Using now the formulae Vekua-Bitsadze for the homogenous matrix equations (6) we obtain the following complex representation of the general solutions

$$V = 2Re\{\varphi(z) + \frac{A}{4}\int\limits_{z_0}^{z}\int\limits_{\overline{z}_0}^{\overline{z}} \Lambda(t,\overline{t})R(z,\overline{z},t,\overline{t})\varphi(t)dtd\overline{t}\},$$

$$\Omega = 2Re\{f(z) + \frac{B}{4}\int\limits_{z_0}^{z}\int\limits_{\overline{z}_0}^{\overline{z}} \Lambda(t,\overline{t})r(z,\overline{z},t,\overline{t})f(t)dtd\overline{t}\},$$

where R and r are the Riemann's matrix functions of the equations (6), $\varphi(z)$ and $f(z)$ are holomorphic column-matrices:

$$\varphi(z) = (\varphi_2(z), \cdots, \varphi_N(z), \varphi_{N+1}(z), \cdots \varphi_{2N}(z))^T, \quad f(z) = (f_1(z), \cdots, f_N(z))^T.$$

Then particular solutions of the matrix equations (6) have the form

$$\hat{V}(z,\overline{z}) = \frac{1}{4}\int\limits_{z_0}^{z}\int\limits_{\overline{z}_0}^{\overline{z}} \Lambda(t,\overline{t})\hat{R}(z,\overline{z},t,\overline{t})X(t,\overline{t})dtd\overline{t},$$

$$\hat{\Omega}(z,\overline{z}) = \frac{1}{4}\int\limits_{z_0}^{z}\int\limits_{\overline{z}_0}^{\overline{z}} \Lambda(t,\overline{t})\hat{r}(z,\overline{z},t,\overline{t})Y(t,\overline{t})dtd\overline{t}.$$

where $\hat{R}$ and $\hat{r}$ are so-called elementary solutions of (6).

REFERENCES

1. Vekua I.N. Shell Theory: General methods of construction. *Pitman Advanced Publishing Program, Boston-London-Melburne*, 1985, 287 p.

2. Goldenveizer A.L. Theory of elastic thin shells. (Russian) *Moscow, Nauka,* 1976, 512 p.

3. Bitsadze A. The value boundary value problem for the elliptical system equations of the second order. (Russian) *Moscow, Nauka,* 1966, 203 p.

4. Meunargia T.V. A Small-parameter method for I. Vekua's nonlinear and non-shallow shells. *Proceeding of the IUTAM Symposium, Springer Science* 2008, 155-166.

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