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ESSENTIALLY NONLINEAR GENERALIZED DIFFERENTIAL EQUATION OF EMDEN-FOWLER TYPE WITH DELAY ARGUMENT

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Abstract. In the paper the following differential equation

$$u^{(n)}(t) + p(t)\left|u(\sigma(t))\right|^{\mu(t)} \operatorname{sign} u(\sigma(t)) = 0, \tag{0.1}$$

is considered, where $p \in L_{\text{loc}}(R_+; R)$, $\sigma \in C(R_+, R_+)$, $\sigma(t) \le t$ for $t \in R_+$, $\lim\limits_{t\to+\infty} \sigma(t) = +\infty$ and $\mu \in C(R_+, (1, +\infty))$. New sufficient conditions, the Eq. (0.1) to have Property **A** or Property **B**, are established.

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1. Introduction

The work concerns the study of oscillatory properties of the differential equation

$$u^{(n)}(t) + p(t)\left|u(\sigma(t))\right|^{\mu(t)} \operatorname{sign} u(\sigma(t)) = 0, \tag{1.1}$$

where $p \in L_{\text{loc}}(R_+; R)$, $\mu \in C(R_+; (1, +\infty))$, $\sigma \in C(R_+; R_+)$, $\sigma(t) \le t$, $\lim\limits_{t\to+\infty} \sigma(t) = +\infty$ and

$$\mu(t) \ge \lambda > 1 \quad \text{for} \quad t \in R_+. \tag{1.2}$$

It will always be assumed that either the condition

$$p(t) \ge 0 \quad \text{for} \quad t \in R_+ \tag{1.3}$$

or the condition

$$p(t) \le 0 \quad \text{for} \quad t \in R_+ \tag{1.4}$$

is fulfilled.

Let $t_0 \in R$. A function $u : [t_0, +\infty) \to R$ is said to be a proper solution of Eq. (1.1) if it is locally absolutely continuous together with its derivatives up to order $n-1$ inclusive,

$$\sup\left\{|u(s)| : s \in [t, +\infty)\right\} > 0 \quad \text{for} \quad t \ge t_0$$

and there exists a function $\overline{u} \in C(R_+ : R)$ such that $\overline{u}(t) \equiv u(t)$ on $[t_0, +\infty)$ and the equality

$$\overline{u}^{\,(n)}(t) + p(t)\left|\overline{u}(\sigma(t))\right|^{\mu(t)} \operatorname{sign} \overline{u}(\sigma(t)) = 0$$

holds for $t \in [t_0, +\infty)$. A proper solution $u : [t_0, +\infty) \to R$ of Eq. (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution u is said to be nonoscillatory.

Definition 1.1. We say that the Eq. (1.1) has property **A** if any of its proper solutions is oscillatory when n is even and is either is oscillatory or satisfies

$$\left|u^{(i)}(t)\right| \downarrow 0 \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.5}$$

when n is odd.

Definition 1.2. We say that the Eq. (1.1) has property **B** if any of its proper solutions either is oscillatory or satisfies (1.5) or

$$\left|u^{(i)}(t)\right| \uparrow +\infty \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.6}$$

when n is even and either is oscillatory or satisfies (1.6) when n is odd.

A number of survey papers and monographs have been devoted to various aspects of oscillation of nonlinear differential equations with a delayed argument (see, for example, [1–3]).

2. Differential equation with property A and B

Theorem 2.1. *Let the conditions* (1.2), (1.3) ((1.4)) *be fulfilled,*

$$\liminf_{t\to+\infty} \frac{(\sigma(t))^{\mu(t)}}{t} > 0, \quad \limsup_{t\to+\infty} \mu(t) < +\infty \tag{2.1}$$

and for some $\varepsilon > 0$

$$\int_1^{+\infty} t^{n-1} \left(\frac{\sigma(t)}{t}\right)^{1+\varepsilon} |p(t)|\, dt = +\infty. \tag{2.2}$$

Then Eq. (1.1) *has Property* **A** (**B**).

Remark 2.1. The condition (2.2) cannot be replaced to the condition

$$\int_1^{+\infty} t^{n-2}\, \sigma(t)\, |p(t)|\, dt = +\infty.$$

Corollary 2.1. Let the conditions (1.2), (1.3) ((1.4)) be fulfilled,

$$\liminf_{t\to+\infty} \frac{\sigma(t)}{t} > 0, \quad \limsup_{t\to+\infty} \mu(t) < +\infty.$$

Then the condition

$$\int_0^{+\infty} t^{n-1}\, |p(t)|\, dt = +\infty \tag{2.3}$$

is necessary and sufficient for the Eq. (1.1) to have Property **A** (**B**).

Theorem 2.2. *Let the conditions* (1.2), (1.3) *be fulfilled,*

$$\limsup_{t\to+\infty} \frac{(\sigma(t))^{\mu(t)}}{t} < +\infty, \quad \limsup_{t\to+\infty} \mu(t) < +\infty \tag{2.4}$$

and for some $\varepsilon > 0$

$$\int_1^{+\infty} (\sigma(t))^{1+(n-2)\mu(t)} \left(\frac{\sigma(t)}{t}\right)^{\varepsilon} p(t)\, dt = +\infty.$$

Then Eq. (1.1) *has Property* **A**.

Theorem 2.3. *Let the conditions* (1.2), (1.4) (2.4) *be fulfilled and for some* $\varepsilon > 0$

$$\int_1^{+\infty} t(\sigma(t))^{(n-3)\mu(t)+1} \Big(\frac{\sigma(t)}{t}\Big)^{\varepsilon} |p(t)|\, dt = +\infty.$$

Then Eq. (1.1) *has Property* **B**.

Theorem 2.4. *Let n be odd (n be even), the conditions* (1.2), (1.3), ((1.4)) *be fulfilled and for some* $\varepsilon > 0$

$$\liminf_{t\to+\infty} \frac{(\sigma(t))^{1+\varepsilon+\mu(t)}}{t^{2+\varepsilon}} > 0.$$

Then the condition (2.3) *is necessary and sufficient for Eq.* (1.1) *to have Property* **A** (**B**).

Theorem 2.5. *Let the conditions* (1.2), (1.4) *and the second condition of* (2.4) *be fulfilled and for some* $\varepsilon > 0$

$$\limsup_{t\to+\infty} \frac{(\sigma(t))^{2\mu(t)-1-\varepsilon}}{t^{1-\varepsilon}} < +\infty.$$

Then the condition

$$\int_1^{+\infty} (\sigma(t))^{(n-1)\mu(t)} |p(t)|\, dt = +\infty$$

is necessary and sufficient for Eq. (1.1) *to have Property* **B**.

To simplify things, consider the Emden-Fowler type differential equation

$$u^{(n)}(t) + p(t)\,\big|u(t^{\alpha})\big|^{\lambda} \operatorname{sign} u(t^{\alpha}) = 0, \tag{2.5}$$

where $\alpha \in (0,1)$, $\lambda \in (1,+\infty)$.

Corollary 2.2. Let n be odd (n be even), the conditions (1.3) ((1.4)) be fulfilled and $\alpha > \frac{2}{1+\lambda}$. Then the condition (2.3) is necessary and sufficient for Eq. (2.5) to have Property **A** (**B**).

Corollary 2.3. Let the conditions (1.4) holds and $\alpha < \frac{1}{2\lambda-1}$. Then the condition

$$\int_0^{+\infty} t^{\alpha\lambda(n-1)} |p(t)|\, dt = +\infty$$

is necessary and sufficient for Eq. (2.5) to have Property **B**.

R E F E R E N C E S

1. Koplatadze R.G. On some properties of the solutions of nonlinear differential equations and retarded argument. (Russian) *Differentsial'nye Uravneniya* **12**, 11 (1976), 1971–1984.

2. Koplatadze R.G., Chanturia T.A. On oscillatory properties of differential equations with a deviating argument. (Russian) *Tbilisi State University Press, Tbilisi,* 1977.

3. Koplatadze R. On oscillatory properties of solutions of functional differential equations. *Mem. Differential Equations Math. Phys.* **3** (1994), 1-179.

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