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ON ONE NON-LINEAR BOUNDARY VALUE PROBLEM FOR HOLOMORPHIC FUNCTIONS

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Abstract. The nonlinear problem for the holomorphic function in a lattice type domain with ellipsoidal cuts is studied. The effective solutions are obtained by means of conformal mapping and integral equation method. Hence, the solution of the Dirichlet problem for the Laplace equation in the rectangular type lattice with elliptical cuts is obtained. The results could be applied to the oxy-symmetrical problems of hydrodynamics and nanomaterials with the rectangular type lattice.

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Oxy-symmetrical problems of hydrodynamics are closely connected with the conformal mappings [4-7]. In this paper we will consider the conformal mapping of the rectangle with the ellipsoidal cuts on the rectangle. This problem is connected with the oxysymmetrical motion of the elliptic body under stress in the narrow channel.

Let in a complex $z = x + iy$ plane the area D represents a rectangle $\{-l \leq x \leq l;\ -h \leq y \leq h\}$ cut along the ellipses with the centers $(-a, 0)$ and $(a, 0)$ given by the formulae

$$\frac{(x+a)^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1}$$

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{2}$$

where a, b, l and h are the definite positive numbers, $0 < b < h,\ 0 < a < l$.

Our purpose is to solve the following problem

Problem 1. To find conformal mapping $f(z)$ of the area D on the rectangle D_0 of the complex plane $w = \xi + i\eta$,

$$D_0 = \{-2l_0 \leq \xi \leq 2l_0;\ \ -h_0 \leq \xi \leq h_0\} \quad (l_0, h_0 > 0),$$

cut along the segment $[-2a_0, 2a_0]$ with the following correspondence of points

$$-l + ih \leftrightarrow -l_0 + ih_0;\ \ -2a \leftrightarrow -2a_0;\ \ 0 \leftrightarrow 0.$$

According to the principle of symmetry we will have $l + ih \leftrightarrow l_0 + ih_0;\ 2a \leftrightarrow 2a_0$.

The function f map the upper part of the left ellipse (1) on the upper side of the segment $[-2a_0, 0]$ and to the bottom part corresponds the another (law) side of the segment $[-2a_0, 0]$. Consequently , to the right ellipse (2) corresponds the upper and law parts of the segment $[0, 2a_0]$ consequently.

We assume that the mapping f is symmetric. So, we can find the mapping of the upper part of the area D $(y \geq 0)$ on the upper part of the rectangle D $(\eta \geq 0)$

Let $z = F(w) = x(\xi,\eta) + iy(\xi,\eta)$ be the inverse mapping. The function $F(w)$ is the solution of the following problem:

Problem 2. In the area D_0 $(\eta \geq 0)$ find a holomorphic function $z = F(w)$ satisfying the following boundary conditions.

$$\begin{aligned}
&\mathrm{Im}F|_{\eta=h_0} = h,\\
&\mathrm{Im}F|_{[-l_0,-2a_0]\cup[2a_{0,l_0}]} = 0,\\
&\mathrm{Im}F|_{\xi=l_0} = -l,\\
&\frac{(\mathrm{Re}F+a)^2}{a^2} + \frac{\mathrm{Im}^2F}{b^2} = 1 \;\; \xi \in [0,2a_0],\\
&\frac{(\mathrm{Re}F-a)^2}{a^2} + \frac{\mathrm{Im}^2F}{b^2} = 1 \;\; \xi \in [0,2a_0].
\end{aligned} \tag{3}$$

The function $F' = \dfrac{\partial x}{\partial \xi} - i\dfrac{\partial x}{\partial \eta}$ is the holomorphic (with the possible exception of the points $(-2a_0,0)$, $(0,0)$, $(0,2a_0)$ and according to (3) satisfy the following conditions

$$\mathrm{Im}F' = 0,\; \xi \in [-l_0, 2a_0] \cup [2a_0, l_0] \text{ and } \eta = h_0, \tag{4}$$

$$\mathrm{Re}F'|_{\xi=-l_0,\,\xi=l_0} = 0, \tag{5}$$

$$\mathrm{Im}F' = |_{[-2a_0,2a_0]} = y_0'(\xi). \tag{6}$$

where $y_0(\xi)$ is the unknown function. By the conditions (4) and (5) we can continue $F'(\xi,\eta)$ analytically in the stripe $0 \leq \eta \leq h_0$ and using Villas formula for the stripe [1,2,4,5,7] and (6) we obtain

$$F'(w) = \frac{1}{\pi}\int\limits_{-2a_0}^{a_0} y_0'[\zeta(t-w) - \zeta(t)]dt + C, \tag{7}$$

where ζ is the Weierstrass "ζ- function" for the periods $2l_0$ and $2ih_0$, C is the definite constant.

Taking into the account symmetricity of $y(\xi)$ from (7) we obtain

$$F'(w) = -\frac{1}{\pi}\int\limits_{0}^{2a_0} y_0'[\zeta(t-w) + \zeta(t+w) - 2\zeta(t)]dt + C. \tag{8}$$

After integration (8) takes the form

$$F(w) = \frac{1}{\pi}\int\limits_{0}^{2a_0} y_0[\zeta(t+w) - \zeta(t-w)]dt + Cw. \tag{9}$$

If we consider the limiting values of (9) for $\eta \to 0,\ \xi \in [-2a_0, 2a_0]$ we obtain

$$
\begin{aligned}
&a - a\sqrt{1-\frac{y_0^2}{b^2}} = \frac{1}{\pi}\int\limits_0^{2a_0} y_0 K(t,\xi)dt + C\xi,\ \ \xi \in [0, a_0],\\
&a + a\sqrt{1-\frac{y_0^2}{b^2}} = \frac{1}{\pi}\int\limits_0^{2a_0} y_0 K(t,\xi)dt + C\xi,\ \ \xi \in [a_0, 2a_0],\\
&K(t,\xi) = \zeta(t-\xi) - \zeta(t+\xi).
\end{aligned}
\tag{10}
$$

Two equations of (10) are the non-linear integral equations with respect to $y_0(\xi)$. If we put $\xi = 2a_0 - \xi_0$ into second equation, we obtain

$$
a + a\sqrt{1-\frac{y_0^2}{b^2}} = \frac{1}{\pi}\int\limits_0^{2a_0} y_0 K(t,\xi_0)dt + C(2a_0-\xi_0),\ \ \xi_0 \in [0, a_0]. \tag{11}
$$

Adding the first equation of (10) and (11) after simple transformations we obtain the linear integral equation

$$
\frac{1}{\pi}\int\limits_0^{a} y_0[\zeta(t+\xi-4a_0) - \zeta(t+\xi)]dt = C_0^*, \tag{12}
$$

where C_0^* is the definite constant.

(12) is the integral equation with the Weierstrass Kernel according to the results of [3] the solution of this equation is representable by the formula.

$$
y_0 = \mathrm{Re}\Phi(w) = \mathrm{Re}\, C_0 \int\limits_{a_0}^{w} \frac{\sqrt{\sigma(t-4a_0)\sigma(t+4a_0)\sigma(t-3a_0)\sigma(t+3a_0)}}{\sigma(t)\sqrt{\sigma(t-a_0)\sigma(t+a_0)}} + b, \tag{13}
$$

where C_0 is the definite constant.

Putting (13) into (9) we obtain the solution of the problem 2 and consequently of the problem 1.

Analogously the conformal mapping of the stripe D_1 with one symmetrical ellipsoidal cuts are obtained.

Problem 3. Find the conformal mapping $z = F_2(w)$ of the D_0 on the stripe D_1 with the ellipsoidal cuts $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, with the following correspondence of points $-l_0 + ih_0 \leftrightarrow -l + ih;\ -a_0 \leftrightarrow -a;\ 0 \leftrightarrow b$

The solution of the problem 3 is given by the formula

$$
F_1(w) = \frac{1}{\pi}\int\limits_0^{a_0} y_1[\zeta(t+w) - \zeta(t-w)]dt + (ib)
$$

where $y_1 = C_1 \displaystyle\int\limits_0^{\xi} \frac{\sigma(t)}{\sqrt{\sigma(a_0-t)\sigma(a_0+t)}}\, dt + b, \quad \xi \in [0, a_0],$

C_1 is the definite constant.

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