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EQUATIONS WITH ORDER DEGENERATION AND AXIALLY SYMMETRIC SOLUTIONS OF ELLIPTIC EQUATIONS

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Abstract. The paper deals with a question of the relation between axially symmetric solutions of the second order elliptic equations of $p \geq 3$ variables and degenerate partial differential equations of two variables. Using explicit solutions to some boundary value problems for a degenerate partial differential equations of two variables, some problems for, in general, singular partial differential equations of $p \geq 3$ variables is solved in the explicit form.

Keywords and phrases: Axially symmetric solutions, elliptic equations, singular partial differential equations, degenerate partial differential equations.

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Let us consider axially symmetric with respect to the axis x_1 solutions $u(x) = u(x_1, y)$, $y := \sqrt{x_2^2 + \cdots + x_p^2}$, to the elliptic equation of p independent variables of the canonical form

$$u_{,ii}(x) + a_i(x)u_{,i}(x) + b(x)u(x) = 0, \tag{1}$$

where $x := (x_1, \cdots, x_p) \in \mathbb{R}^p$, and the usual differentiation and summation conventions are used.

Clearly, we have

$$u_{,i} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial x_i} = \frac{\partial u}{\partial y}\frac{x_i}{y}, \quad i = 2, \cdots, p, \tag{2}$$

$$u_{,\underline{ii}} = \frac{\partial^2 u}{\partial y^2}\frac{x_i^2}{y^2} + \frac{\partial u}{\partial y}\left(\frac{1}{y} - \frac{x_i^2}{y^3}\right), \quad i = 2, \cdots, p, \tag{3}$$

where hyphen under repeated indeces means that we do not sum.

Substituting (2) and (3) in (1), we obtain

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial y^2} + a_1\frac{\partial u}{\partial x_1} + \frac{1}{y}\left(p - 2 + \sum_{i=2}^{p} a_i x_i\right)\frac{\partial u}{\partial y} + bu = 0. \tag{4}$$

If

$$p - 2 + \sum_{i=2}^{p} a_i x_i = a(x_1, y) \tag{5}$$

and

$$a(x_1, y) = O(y^\alpha), \quad y \to 0+, \quad 0 \leq \alpha < 1, \tag{6}$$

then (4) will be singular partial differential equation with singularity by $y = 0$. In the last case, multiplaying equation (4) by y, it becomes a degenerate partial differential equation with the order degeneration.

If

$$a(x_1, y) = O(y^\alpha), \quad y \to 0+, \quad \alpha \geq 1,$$

then equation (4) is not singular one.

So, under the conditions (5), (6) axially symmetric solutions of equation (1), satisfy singular differential equation (4). In particular, in the case of the Laplace equation, i.e., when $a_i \equiv 0$, $i = \overline{1,p}$, $b \equiv 0$, symmetric harmonic functions of p independent variables $x_1, \ldots, x_p$ satisfy with respect to x_1 and y the following equation with the order degeneration

$$y\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial y^2}\right) + (p-2)\frac{\partial u}{\partial y} = 0. \tag{7}$$

Let us consider more general than (7) equation

$$y\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial y^2}\right) + a\frac{\partial u}{\partial x_1} + (p-2)\frac{\partial u}{\partial y} = 0, \tag{8}$$

where a is an arbitrary real constant. The corresponding to (8) (compare with (4)) equation (1) has the form

$$u_{,ii}(x) + \frac{a}{\sqrt{x_2^2 + \cdots + x_p^2}} u_{,1}(x) = 0. \tag{9}$$

Equation (9) is a singular partial differential equation with singularity on the axis x_1, which can be rewritten as an equation with the order degeneration on the axis x_1:

$$\left(x_2^2 + \cdots + x_p^2\right)^{\frac{1}{2}}\left(u_{,11} + \cdots + u_{,pp}\right) + au_{,1} = 0. \tag{10}$$

In $\mathbb{R}^p$, $p \geq 3$, axially symmetric with respect to x_1 solutions

$$u \in C^2\Big(\mathbb{R}^p \backslash \Big(\mathbb{R}^1 \times \underbrace{\{0\} \times \cdots \times \{0\}}_{(p-1)-times}\Big)\Big)$$

of equation (10) will be solutions to equation (7) with

$$y := \sqrt{x_2^2 + \cdots + x_p^2}. \tag{11}$$

From Theorem 2.6 and Theorem 2.8 (see [1], pp. 42, 51) there follows the following assertion:

1. the expression

$$u = M^{-1}(a, p-2, m) \int\limits_{-\infty}^{+\infty} f(\xi) e^{a\theta} \rho^{2-p} d\xi$$

where

$$M(a, b, m) := y^{b+m-1} \int\limits_{-\infty}^{\infty} \frac{\partial^m e^{a\theta} \rho^{-b}}{\partial y^m} d\xi,$$

$$\theta = \text{arc ctg}\frac{x - \xi}{y}, \quad \rho = [(x-\xi)^2 + y^2]^{1/2},$$

with (11) and $p \in \mathbb{N}\backslash\{1,2\}$,

$$m \in \begin{cases} \mathbb{N}^0, & p > 3, \\ \mathbb{N}, & p = 3, \end{cases} \qquad m > 3 - p,$$

($\mathbb{N}$ is the set of natural numbers, $\mathbb{N}^0 := \mathbb{N} \cup \{0\}$) represents a unique solution of the problem:

Find in

$$\mathbb{R}^p\backslash\Big(\mathbb{R}^1 \times \underbrace{\{0\} \times \cdots \{0\}}_{(p-1)-times}\Big)$$

axially symmetric with respect to x_1 solution

$$u \in C^2\Big(\mathbb{R}^p\backslash\Big(\mathbb{R}^1 \times \underbrace{\{0\} \times \cdots \{0\}}_{(p-1)-times}\Big)\Big)$$

of equation (10) satisfying the following conditions:
(i) on the axis of symmetry x_1

$$\lim_{(x_2^2+\cdots+x_p^2)\to 0} (x_2^2 + \cdots + x_p^2)^{\frac{p+m-3}{2}} \frac{\partial^m u(x_1,\ldots,x_p)}{\partial\sqrt{x_2^2+\cdots+x_p^2}^{\,m}} = f(x_1), \quad x_1 \in \mathbb{R}^1,$$

where f is a continuous function; if $p = 3$ we additionally demand

$$f(x_1) = O(|x_1|^{-\alpha}), \quad |x_1| \to +\infty, \quad \alpha > 0, \tag{12}$$

$$\lim_{(x_2^2+x_3^2)\to 0} \int\limits_{-\infty}^{+\infty} (x_2^2 + x_3^2)^{\frac{1}{2}} u(x_1,x_2,x_3) \frac{\partial u(x_1,x_2,x_3)}{\partial\sqrt{x_2^2+x_3^2}} dx_1 = 0, \tag{13}$$

when

$$\lim_{(x_2^2+x_3^2)\to 0} \left[\frac{1}{2}\ln(x_2^2+x_3^2)\right]^{-1} u(x_1,x_2,x_3) = 0, \quad x_1 \in \mathbb{R}^1; \tag{14}$$

(ii) at infinity for $p \geq 4$

$$u(x_1,\ldots,x_p) = \begin{cases} O\Big((x_2^2+\cdots+x_p^2)^{\frac{3-p}{2}}\Big), \quad x_1^2+\cdots+x_p^2 \to +\infty, \\ \text{when either } \ a \in \mathbb{R}^1, \ \ p \in \mathbb{N}\backslash\{1,2,3,4\}, \ \ \text{or} \ \ a = 0, \ \ p = 4; \\ o\Big((x_2^2+\cdots+x_p^2)^{\frac{3-p}{2}}\Big), \quad x_1^2+\cdots+x_p^2 \to +\infty, \\ \text{when } \ a \neq 0, \ \ p = 4, \end{cases}$$

for $p = 3$

$$u(x_1,x_2,x_3) = O((x_1^2+x_2^2+x_3^2)^{-1}), \quad x_1^2+x_2^2+x_3^2 \to +\infty, \tag{15}$$

$$\frac{\partial u(x_1,x_2,x_3)}{\partial x_1}, \ \frac{\partial u(x_1,x_2,x_3)}{\partial\sqrt{x_2^2+x_3^2}} = O((x_1^2+x_2^2+x_3^2)^{-2}), \quad x_1^2+x_2^2+x_3^2 \to +\infty, \tag{16}$$

2. The expression

$$u = (1 + e^{a\pi})^{-1} \int\limits_{-\infty}^{+\infty} f(\xi) e^{a\theta} \rho^{-1} d\xi,$$

with (11) and $p = 3$, $m = 0$ represents a unique solution of the problem:

Find in

$$\mathbb{R}^3 \backslash (\mathbb{R}^1 \times \{0\} \times \{0\})$$

a symmetric with respect to axis x_1 solution $u \in C^2(\mathbb{R}^3 \backslash (\mathbb{R}^1 \times \{0\} \times \{0\}))$ to the equation

$$(x_2^2 + x_3^2)^{\frac{1}{2}} (u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}) + a u_{x_1} = 0,$$

satisfying the following conditions:
(i) on the axis of symmetry x_1

$$\lim_{(x_2^2 + x_3^2) \to 0} \left[\frac{1}{2} \ln(x_2^2 + x_3^2) \right]^{-1} u(x_1, x_2, x_3) = f(x_1), \quad x_1 \in \mathbb{R}^1,$$

where f is a continuous function, satisfying (12), and (13) is fulfilled for (14);
(ii) at infinity (15) and (16) are fulfilled.

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