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I. VEKUA'S METHOD FOR THE GEOMETRICALLY NONLINEAR AND NON-SHALLOW CYLINDRICAL SHELLS

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Abstract. In the present paper we consider the geometrically nonlinear and non-shallow cylindrical shells. By means of I. Vekua method the system of equilibrium equations in two variables is obtained. Using complex variable functions and the method of the small parameter approximate solutions are constructed for $N = 0$ in the hierarchy by I. Vekua. Concrete problem is solved, when the components of the external force are constants.

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We consider the system of equilibrium equations of the two-dimensional geometrically nonlinear and non-shallow cylindrical shells which is obtained from the three-dimensional problems of the theory of elasticity for isotropic and homogeneous shell by the method of I. Vekua [1],[5].

The system of equilibrium equations of the two-dimensional geometrically nonlinear shallow cylindrical shells may be written in the following form:

$$\begin{aligned}
&\partial_\alpha \overset{(0)}{\sigma}_{\alpha 1} + \varepsilon \overset{(0)}{\sigma}_{13} + \overset{(0)}{F}_1 = 0,\\
&\partial_\alpha \overset{(0)}{\sigma}_{\alpha 2} + \overset{(0)}{F}_2 = 0,\\
&\partial_\alpha \overset{(0)}{\sigma}_{\alpha 3} - \varepsilon \overset{(0)}{\sigma}_{11} + \overset{(0)}{F}_3 = 0,
\end{aligned} \tag{1}$$

$$\overset{(0)}{\boldsymbol{F}} = \overset{(0)}{\boldsymbol{\Phi}} + \frac{1}{2h}\left[(1+\varepsilon)\overset{(+)}{\boldsymbol{\sigma}}_3 - (1-\varepsilon)\overset{(-)}{\boldsymbol{\sigma}}_3\right],$$

$$\left(\overset{(0)}{\sigma}_{ij},\ \overset{(0)}{\boldsymbol{\Phi}}\right) = \frac{1}{2h}\int\limits_{-h}^{h}\left(1+\frac{x_3}{R}\right)(\sigma_{ij},\ \boldsymbol{\Phi})dx_3.$$

$$\boldsymbol{\sigma}_3(x_1, x_2, \pm h) = \overset{(\pm)}{\boldsymbol{\sigma}}_3,$$

where $\boldsymbol{\Phi}$ is an external force, σ_{ij} are covariant components of the stress tensor, x_1 and x_2 are isometric coordinates on the cylindrical surface, x_3 is the thickness coordinate, $\varepsilon = h/R$ is the small parameter, h is the semi-thickness of the shell, R is the radius of the middle surface of the cylinder.

Hook's law have the form:

$$\overset{(0)}{\boldsymbol{\sigma}}_1 = \Big(1+\frac{\varepsilon^2}{3}+\cdots\Big)\Big[(\lambda+\mu)(\boldsymbol{r}_1\partial_1\boldsymbol{u})\boldsymbol{r}_1+\mu\partial_1\boldsymbol{u}+\lambda(\boldsymbol{r}_2\partial_2\boldsymbol{u})\partial_1\boldsymbol{u}+\mu(\boldsymbol{r}_2\partial_1\boldsymbol{u})\partial_2\boldsymbol{u}$$
$$+\mu(\partial_1\boldsymbol{u}\partial_2\boldsymbol{u})\boldsymbol{r}_2+\frac{\lambda}{2}(\partial_2\boldsymbol{u}\partial_2\boldsymbol{u})\partial_1\boldsymbol{u}+\mu(\partial_1\boldsymbol{u}\partial_2\boldsymbol{u})\partial_2\boldsymbol{u}\Big]+\lambda(\boldsymbol{r}_2\partial_2\boldsymbol{u})\boldsymbol{r}_1+\mu(\boldsymbol{r}_1\partial_2\boldsymbol{u})\boldsymbol{r}_2$$
$$+\frac{\lambda}{2}(\partial_2\boldsymbol{u}\partial_2\boldsymbol{u})\boldsymbol{r}_1+\mu(\boldsymbol{r}_1\partial_2\boldsymbol{u})\partial_2\boldsymbol{u}+(1+\varepsilon^2+\cdots)(\lambda+2\mu)\Big[(\boldsymbol{r}_1\partial_1\boldsymbol{u})\partial_1\boldsymbol{u}$$
$$+\frac{1}{2}(\partial_1\boldsymbol{u}\partial_1\boldsymbol{u})\boldsymbol{r}_1\Big]+\Big(1+2\varepsilon^2+\cdots\Big)\frac{\lambda+2\mu}{2}(\partial_1\boldsymbol{u}\partial_1\boldsymbol{u})\partial_1\boldsymbol{u},$$
$$\overset{(0)}{\boldsymbol{\sigma}}_2 = \lambda(\boldsymbol{r}_\beta\partial_\beta\boldsymbol{u})\boldsymbol{r}_2+\mu(\boldsymbol{r}_2\partial_\beta\boldsymbol{u})\boldsymbol{r}_\beta+\mu\partial_2\boldsymbol{u}+(\lambda+2\mu)(\boldsymbol{r}_2\partial_2\boldsymbol{u})\partial_2\boldsymbol{u}+\lambda(\boldsymbol{r}_1\partial_1\boldsymbol{u})\partial_2\boldsymbol{u} \qquad (2)$$
$$+\mu(\boldsymbol{r}_1\partial_2\boldsymbol{u})\partial_1\boldsymbol{u}+\mu(\partial_1\boldsymbol{u}\partial_2\boldsymbol{u})\boldsymbol{r}_1+\frac{\lambda+2\mu}{2}\Big[(\partial_2\boldsymbol{u}\partial_2\boldsymbol{u})\boldsymbol{r}_2+(\partial_2\boldsymbol{u}\partial_2\boldsymbol{u})\boldsymbol{r}_2\Big]$$
$$+\Big(1+\frac{\varepsilon^2}{3}+\cdots\Big)\Big[\frac{\lambda}{2}(\partial_1\boldsymbol{u}\partial_1\boldsymbol{u})\boldsymbol{r}_2+\mu(\boldsymbol{r}_2\partial_1\boldsymbol{u})\partial_1\boldsymbol{u}+\frac{\lambda}{2}(\partial_1\boldsymbol{u}\partial_1\boldsymbol{u})\partial_2\boldsymbol{u}+2\mu(\partial_1\boldsymbol{u}\partial_2\boldsymbol{u})\partial_1\boldsymbol{u}\Big],$$
$$\overset{(0)}{\boldsymbol{\sigma}}_3 = \lambda\Big[(\boldsymbol{r}_\beta\partial_\beta\boldsymbol{u})+\frac{1}{2}(\partial_2\boldsymbol{u}\partial_2\boldsymbol{u})\Big]\boldsymbol{n}+\mu\Big[(\boldsymbol{n}\partial_\beta\boldsymbol{u})\boldsymbol{r}_\beta+(\boldsymbol{n}\partial_2\boldsymbol{u})\partial_2\boldsymbol{u}\Big]$$
$$+\Big(1+\frac{\varepsilon^2}{3}+\cdots\Big)\Big[\frac{\lambda}{2}(\partial_1\boldsymbol{u}\partial_1\boldsymbol{u})\boldsymbol{n}+\mu(\boldsymbol{n}\partial_1\boldsymbol{u})\partial_1\boldsymbol{u}\Big],$$

where

$$\partial_1\boldsymbol{u}\cdot\partial_1\boldsymbol{u}=\frac{1}{2}\left[(\partial_1u_1)^2+(\partial_1u_2)^2+(\partial_1u_3)^2+\varepsilon^2(u_1^2+u_3^2)\right],$$
$$\partial_2\boldsymbol{u}\cdot\partial_2\boldsymbol{u}=\frac{1}{2}\left[(\partial_2u_1)^2+(\partial_2u_2)^2+(\partial_2u_3)^2\right],$$
$$\partial_1\boldsymbol{u}\cdot\partial_2\boldsymbol{u}=\frac{1}{2}\left[\partial_1u_1\cdot\partial_2u_1+\partial_1u_2\cdot\partial_2u_2+\partial_1u_3\cdot\partial_2u_3+\varepsilon u_3\partial_2u_3-\varepsilon u_1\partial_2u_1\right],$$
$$\boldsymbol{r}_1\partial_1\boldsymbol{u}=\partial_1u_1+\varepsilon u_3,\quad \boldsymbol{n}\partial_1\boldsymbol{u}=\partial_1u_3-\varepsilon u_1$$
$$\boldsymbol{r}_1\partial_2\boldsymbol{u}=\partial_2u_1,\quad \boldsymbol{r}_2\partial_1\boldsymbol{u}=\partial_1u_2,\quad \boldsymbol{r}\partial_2\boldsymbol{u}=\partial_2u_2,\quad \boldsymbol{n}\partial_2\boldsymbol{u}=\partial_2u_3,$$
$$\boldsymbol{u}=\frac{1}{2h}\int_{-h}^{h}\boldsymbol{v}dx_3.$$

Here $\boldsymbol{v}$ is the displacement vector, λ and μ are Lame's constants.

Let us use the method of the small parameter [7]. The same method has been also used for spherical and cylindrical shallow and non-shallow shells [2],[3],[6],[8],[9].

Let us construct the solutions of the form:

$$u_i=\sum_{k=0}^{\infty}\overset{(k)}{u}_i\varepsilon^k, \qquad (3)$$

Formal substitution of (3) into (2) and (1) shows the series (3) may satisfy equations

(1) if the following equations are fulfilled:

$$
\begin{aligned}
\mu\Delta \overset{(k)}{u}_1 + (\lambda+\mu)\partial_1 \overset{(k)}{\theta} &= \overset{(k)}{X}_1,\\
\mu\Delta \overset{(k)}{u}_2 + (\lambda+\mu)\partial_2 \overset{(k)}{\theta} &= \overset{(k)}{X}_2,\\
\mu\Delta \overset{(k)}{u}_3 &= \overset{(k)}{X}_3\\
(k = 1, 2, ...),
\end{aligned}
\tag{4}
$$

where

$$(k = 0, 1, 2, ...; \quad \overset{(k)}{u}_i = 0, \quad \text{if } k < 0;).$$

For each fixed k equations (4) coincide with equations of plane theory of elasticity and Poisson's equation. The right parts of equations (4) are well-known quantities, defined by functions $\overset{(0)}{\boldsymbol{u}}, \overset{(1)}{\boldsymbol{u}}, ..., \overset{(k-1)}{\boldsymbol{u}}$.

The general solutions of this system are written as following [4]:

$$
\begin{cases}
2\mu \overset{(k)}{u}_+ = æ \overset{(k)}{\varphi}(z) - z\overline{\overset{(k)}{\varphi}{}'(z)} - \overline{\overset{(k)}{\psi}(z)} + \overset{(k)}{u}_{+p},\\
2\mu \overset{(k)}{u}_3 = \overset{(k)}{f}(z) + \overline{\overset{(k)}{f}(z)} + \overset{(k)}{u}_{3p},
\end{cases}
$$

$$\left(z = x_1 + ix_2, \quad \overset{(k)}{u}_+ = \overset{(k)}{u}_1 + i\overset{(k)}{u}_2\right)$$

where $æ = \frac{\lambda+3\mu}{\lambda+\mu}$, $\overset{(k)}{\varphi}(z)$, $\overset{(k)}{\psi}(z)$ and $\overset{(k)}{f}(z)$ are any analytic functions of complex variable z, $\overset{(k)}{u}_{+p}$ and $\overset{(k)}{u}_{3p}$ - particular solutions of the system (4):

$$
\begin{aligned}
\overset{(k)}{u}_{+p} &= \frac{\lambda+3\mu}{2(\lambda+2\mu)}\frac{1}{\pi}\iint_S \overset{(k)}{X}_+ ln|\zeta - z| d\xi d\eta - \frac{\lambda+\mu}{4(\lambda+2\mu)}\frac{1}{\pi}\iint_S \frac{\zeta - z}{\bar\zeta - \bar z}\overline{\overset{(k)}{X}_+} d\xi d\eta,\\
\overset{(k)}{u}_{+3} &= -\frac{1}{\pi}\iint_S \overset{(k)}{X}_3 ln|\zeta - z| d\xi d\eta.
\end{aligned}
$$

Let us solve the problem when the middle surface of the body after developing on the plane, is The circle with the radius r_0 and consider the concrete problem, when the components of external force are constant $\overset{(0)}{F}_i = P_i = const$. The boundary conditions are:

$$u_r = 0, \quad u_\theta = 0, \quad u_3 = 0, \quad (z = re^{i\theta}, \quad |z| = r_0).$$

For $\overset{(1)}{u}_+$ and $\overset{(1)}{u}_3$ we have:

$$
\begin{aligned}
2\mu \overset{(1)}{u}_+ &= \frac{\mu}{\lambda+3\mu}(z\bar z - r_0^2)P_+,\\
2\mu \overset{(1)}{u}_3 &= (z\bar z - r_0^2)\frac{P_3}{2}.
\end{aligned}
$$

The system of equilibrium equations, for the approximation $k=2$, are:

$$\mu\Delta \overset{(2)}{u}_{+} + 2(\lambda+\mu)\partial_{\bar z} \overset{(2)}{\Theta} = A_1 z + A_2 \bar z, \tag{5}$$

$$\Delta \overset{(2)}{u}_{3} = Bz + \bar B \bar z, \tag{6}$$

where

$$\begin{aligned} A_1 &= -\frac{2\lambda+3\mu}{4\mu}P_3 - \frac{\lambda+\mu}{4\mu^2}P_3^2 - \frac{(\lambda+\mu)(P_1^2+P_2^2)}{(\lambda+3\mu)^2}, \\ A_2 &= -\frac{3P_3}{4} - \frac{P_3^2}{4\mu} - \frac{\mu(P_1^2+P_2^2)}{(\lambda+3\mu)^2}, \\ B &= -\frac{2\lambda+3\mu}{4\mu}P_3 - \frac{\lambda+\mu}{4\mu^2}P_3^2 - \frac{(\lambda+\mu)(P_1^2+P_2^2)}{(\lambda+3\mu)^2}, \end{aligned}$$

The general solutions of systems (5) end (6) are written in the following form

$$2\mu \overset{(2)}{u}_{+} = æ \overset{(2)}{\varphi}(z) - z\overline{\overset{(2)}{\varphi}{}'(z)} - \overline{\overset{(2)}{\psi}(z)} + C_1 z^2 \bar z + C_2 z \bar z^2 + C_3 z^3, \tag{7}$$

$$2\mu \overset{(2)}{u}_{3} = \overset{(2)}{f}(z) + \overline{\overset{(2)}{f}(z)} + Dz^2\bar z + \bar D z \bar z^2, \tag{8}$$

where

$$\begin{aligned} C_1 &= \frac{\mu A_1}{4(\lambda+2\mu)}, \qquad C_2 = \frac{\lambda+3\mu}{8(\lambda+2\mu)}A_2, \\ C_3 &= -\frac{\lambda+\mu}{24(\lambda+2\mu)}A_2, \qquad D = \frac{B}{4}. \end{aligned}$$

Boundary conditions are

$$\overset{(2)}{u}_{r} + i \overset{(2)}{u}_{\vartheta} = 0, \qquad \overset{(2)}{u}_{3} = 0, \qquad |\, z \,| = r_0. \tag{9}$$

Let us introduce the functions $\overset{(2)}{\varphi}(z)$, $\overset{(2)}{\psi}(z)$and $\overset{(2)}{f}(z)$ by the series

$$\overset{(2)}{\varphi}(z) = \sum_{n=1}^{\infty} \alpha_n z^n, \quad \overset{(2)}{\psi}(z) = \sum_{n=0}^{\infty} \beta_n z^n, \quad \overset{(2)}{f}(z) = \sum_{0}^{\infty} \gamma_n z^n. \tag{10}$$

By substituting (7), (8) into (9), (10) we obtain

$$\begin{aligned} \alpha_1 &= \frac{\lambda+\mu}{2\mu} r_0^2 C_1, \qquad \alpha_3 = \frac{\lambda+\mu}{\lambda+3\mu}C_3, \\ \beta_1 &= r_0^2 C_2 + \frac{2(\lambda+\mu)}{\lambda+3\mu} r_0^2 C_3, \qquad \gamma_1 = -r_0^2 D. \end{aligned}$$

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