Seminar of I. Vekua Institute of Applied Mathematics REPORTS, Vol. 35, 2009

FINANCIAL MARKETS WITH DISORDERS. OPTIMAL MARTINGALE MEASURES FOR TRINOMIAL SCHEME

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Abstract. In the paper we consider a trinomial scheme with two random disorder moments, which we propose as stock price evolution model, and find entropy minimal martingale measure for one special class of martingale measures.

Keywords and phrases: Trinomial scheme, random disorder moment, martingale measure, relative entropy.

AMS subject classification (2000): 60G42; 60G70; 62P05; 91B26; 94A17.

On the probability space $(\Omega, \mathcal{F}, P)$ consider a real valued stochastic process with discrete time $S = (S_n), n = 0, 1, ..., N$,

$$S_n = S_{n-1}(1 + \rho_n), \tag{1}$$

where $S_0 > 0$ is deterministic and $(\rho_n), n = 1, 2, ..., N$, is the following sequence of random variables:

$$\rho_n = \rho_n^{(1)} I(n < \theta_1) + \rho_n^{(2)} I(\theta \leq n < \theta_1 + \theta_2) + \rho_n^{(3)} I(n \geq \theta_1 + \theta_2). \tag{2}$$

Here $\rho^(i) = (\rho_n^{(i)})$ $(n = 1, 2, ..., N; i = 1, 2, 3)$ are sequences of independent identically distributed random variables that take only three values $a_i, b_i, c_i, a_i < b_i < c_i$ and $-1 < a_i < 0 < c_i$ with probabilities $p_i, q_i, r_i, p_i + q_i + r_i = 1, i = 1, 2, 3$, and θ_1, θ_2 are random variables with values from the sets $\{0, 1, ..., N\}$ and $\{0, 1, ..., N - \theta_1\}$ respectively. Assume, that the distribution $P(\theta_1 = k, \theta_2 = l)$ is known.

$I(A)$ is the indicator of $A \in F$. $\rho^{(1)}, \rho^{(2)}, \rho^{(3)}$ are jointly independent of each other and they are independent of θ_1, θ_2.

It is clear, that until the random moment θ_1 we have one process $S_n^{(1)}$ with the return $\rho_n^{(1)}$, then until the random moment $\theta_1 + \theta_2$ we have the process $S_n^{(2)}$ with the return $\rho_n^{(2)}$ and after $\theta_1 + \theta_2$ we have the process $S_n^{(3)}$ with the return $\rho_n^{(3)}$. Such process described by (1), (2) we call the trinomial scheme with disorder and propose as a model of stock price evolution.

Consider a class of equivalent to P measures $\tilde{P}$ with Radon-Nycodim derivative of following form

$$\begin{aligned} \tfrac{d\tilde{P}}{dP}(\omega) = Z_N(\omega) = \textstyle\prod_{n=1}^{N} [\xi_n^{(1)} I(n < \theta_1) + \xi_n^{(2)} I(\theta_1 \leq n < \theta_1 + \theta_2) \\ + \xi_n^{(3)} I(n > \theta_1 + \theta_2)], \end{aligned} \tag{3}$$

where

$$\xi_n^{(i)} = \frac{\tilde{p}_i}{p_i} I(\rho_n^{(i)} = a_i) + \frac{\tilde{q}_i}{q_i} I(\rho_n^{(i)} = b_i) + \frac{\tilde{r}_i}{r_i} I(\rho_n^{(i)} = c_i), i = 1, 2, 3.$$

Here $\tilde{p}_i, \tilde{q}_i, \tilde{r}_i$ are such positive constants, that

$$\tilde{p}_i + \tilde{q}_i + \tilde{r}_i = 1, i = 1, 2, 3.$$

Now we find the relative entropy

$$I(\tilde{P}, P) = E_P \left[\frac{d\tilde{P}}{dP} \ln \frac{d\tilde{P}}{dP} \right].$$

Direct calculations shows, that

$$I(\tilde{P}, P) = \sum_{k=0}^{N} \sum_{l=0}^{N} P(\theta_1 = k, \theta_2 = l) \Big[(k-1) \Big[\tilde{p}_1 \ln \frac{\tilde{p}_1}{p_1} + \tilde{q}_1 \ln \frac{\tilde{q}_1}{q_1}$$
$$+ \tilde{r}_1 \ln \frac{\tilde{r}_1}{r_1} \Big] + (l+1) \Big[\tilde{p}_2 \ln \frac{\tilde{p}_2}{p_2} + \tilde{q}_2 \ln \frac{\tilde{q}_2}{q_2} + \tilde{r}_2 \ln \frac{\tilde{r}_2}{r_2} \Big]$$
$$+ (N - (k+l)) \Big[\tilde{p}_3 \ln \frac{\tilde{p}_3}{p_3} + \tilde{q}_3 \ln \frac{\tilde{q}_3}{q_3} + \tilde{r}_3 \ln \frac{\tilde{r}_3}{r_3} \Big] \Big].$$

Consider the filtration $G_n = \sigma(\theta_1, \theta_2, \rho_1^{(1)}, \rho_2^{(1)}, ..., \rho_n^{(1)}, \rho_1^{(2)}, ..., \rho_n^{(2)}, \rho_1^{(3)}, ..., \rho_n^{(3)}), n = 1, 2, ..., N, G_0 = \{\varnothing, \Omega\}$. It is easy to see, that

$$Z_n = \prod_{k=1}^{n} \xi_k^{(1)} I(n < \theta_1) + \xi_k^{(2)} I(\theta \le n < \theta_1 + \theta_2) + \xi_k^{(3)} I(n \ge \theta_1 + \theta_2)$$

is the G_n-martingale.

The probability measure $\tilde{P}$ is a martingale measure for S if $\tilde{P} \sim P$ and $S = (S_n, G_n)$ is a martingale.

The martingale condition has the following form $E(\Delta S_n / G_{n-1}) = S_{n-1} \tilde{E}(\rho_n / G_{n-1}) = 0$ or $\tilde{E}(\rho_n / G_{n-1}) = 0$ and

$$0 = \tilde{E}(\rho_n / G_{n-1}) = E(\rho_n Z_N / G_{n-1}) = E[\rho_n E(Z_N / G_n) / G_{n-1}]$$
$$= E(\rho_n Z_N / G_{n-1}) = E[\rho_n Z_{n-1} (\xi_n^{(1)} I(n < \theta_1) + \xi_n^{(2)} I(\theta \le n < \theta_1 + \theta_2)$$
$$+ \xi_n^{(3)} I(n \ge \theta_1 + \theta_2))] = Z_{n-1} [E(\rho_n^{(1)} \xi_n^{(1)}) I(n < \theta_1) + E(\rho_n^{(2)} \xi_n^{(2)})$$
$$\times I(\theta \le n < \theta_1 + \theta_2) + E(\rho_n^{(3)} \xi_n^{(3)}) I(n \ge \theta_1 + \theta_2)],$$

or

$$\begin{aligned} &(a_1 \tilde{p}_1 + b_1 \tilde{q}_1 + c_1 \tilde{r}_1) I(n < \theta_1) + (a_2 \tilde{p}_2 + b_2 \tilde{q}_2 + c_2 \tilde{r}_2) \\ &I(\theta \le n < \theta_1 + \theta_2) + (a_3 \tilde{p}_3 + b_3 \tilde{q}_3 + c_3 \tilde{r}_3) I(n > \theta_1 + \theta_2) = 0. \end{aligned} \quad (4)$$

This martingale condition (4) will be fulfilled if

$$\begin{aligned} &a_1 \tilde{p}_1 + b_1 \tilde{q}_1 + c_1 \tilde{r}_1 = 0 \\ &a_2 \tilde{p}_2 + b_2 \tilde{q}_2 + c_2 \tilde{r}_2 = 0 \\ &a_3 \tilde{p}_3 + b_3 \tilde{q}_3 + c_3 \tilde{r}_3) = 0. \end{aligned} \quad (5)$$

The class of measures defined by (3) represents a class of martingale measures for S under the conditions (5).

Our aim is in this class to construct the measure $\tilde{P}^*$, (entropy minimal martingale measure) which minimizes the relative entropy $I(\tilde{P}, P)$ under the constraints

$$a_1\tilde{p}_1 + b_1\tilde{q}_1 + c_1\tilde{r}_1 = 0, \quad i = 1, 2, 3,$$

$$\tilde{p}_i + \tilde{q}_i + \tilde{r}_i = 0, \quad i = 1, 2, 3.$$

The general problem of finding relative entropy martingale measure is investigated in [3]-[5]. Such problem for the trinomial scheme without disorder we have studied in [1] and with one disorder in [2].

In our case the Lagrangian has the following form

$$\psi = \sum_{k=0}^{N}\sum_{l=0}^{N-k} P(\theta_1 = k, \theta_2 = l)\Big[(k-1)\Big[\tilde{p}_1 \ln\frac{\tilde{p}_1}{p_1} + \tilde{q}_1 \ln\frac{\tilde{q}_1}{q_1}$$
$$+\tilde{r}_1 \ln\frac{\tilde{r}_1}{r_1}\Big] + (l+1)\Big[\tilde{p}_2 \ln\frac{\tilde{p}_2}{p_2} + \tilde{q}_2 \ln\frac{\tilde{q}_2}{q_2} + \tilde{r}_2 \ln\frac{\tilde{r}_2}{r_2}\Big] + (N-(k+l))$$
$$\times\Big[\tilde{p}_3 \ln\frac{\tilde{p}_3}{p_3} + \tilde{q}_3 \ln\frac{\tilde{q}_3}{q_3} + \tilde{r}_3 \ln\frac{\tilde{r}_3}{r_3}\Big]\Big] + \lambda_1(a_1\tilde{p}_1 + b_1\tilde{q}_1 + c_1\tilde{r}_1)$$
$$+\lambda_2(a_2\tilde{p}_2 + b_2\tilde{q}_2 + c_2\tilde{r}_2) + \lambda_3(a_3\tilde{p}_3 + b_3\tilde{q}_3 + c_3\tilde{r}_3)$$
$$+\mu_1(\tilde{p}_1 + \tilde{q}_1 + \tilde{r}_1 - 1) + \mu_2(\tilde{p}_2 + \tilde{q}_2 + \tilde{r}_2 - 1) + \mu_3(\tilde{p}_3 + \tilde{q}_3 + \tilde{r}_3 - 1)$$

and solving this optimization problem under the constraints we obtain the following result:

Theorem. *The Radon-Nykodim derivative of minimal martingale measure $\tilde{P}^*$ has the form*

$$Z_N^*(\omega) = C(\theta)\exp\{-\textstyle\sum_{n=1}^{N}[\tilde{\lambda}_1 I(n<\theta_1) + \tilde{\lambda}_2 I\theta_1 \le n < \theta_1+\theta_2)$$
$$+\tilde{\lambda}_3 I(n \ge \theta_1+\theta_2)]\tfrac{\Delta S_n}{S_{n-1}}\},$$

where $\tilde{\lambda}_i, i = 1, 2, 3,$ are the unique solutions of the following equations

$$a_ip_i\exp\{-a_ix_i\} + b_iq_i\exp\{-b_ix_i\} + c_ir_i\exp\{-c_ix_i\} = 0$$

and

$$C(\theta) = \exp\{-\textstyle\sum_{n=1}^{N}[I(n<\theta_1)\ln D_1 + I(\theta_1 \le n < \theta_1+\theta_2)\ln D_2$$
$$+I(n \ge \theta_1+\theta_2)\ln D_3]\},$$
$$D_i = \frac{1}{p_i\exp\{-\tilde{\lambda}_ia_i\} + q_i\exp\{-\tilde{\lambda}_ib_i\} + r_i\exp\{-\tilde{\lambda}_ic_i\}}, \quad i = 1, 2, 3.$$

R E F E R E N C E S

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Received 24.07.2009; revised 17.09.2009; accepted 14.10.2009.

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