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ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THREE-DIMENSIONAL LINEAR DIFFERENCE SYSTEMS WITH DEVIATING ARGUMENTS

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Abstract. In the paper the following linear differential system

$$x_i'(t) = p_i(t)\, x_{i+1}\big(\tau_{i+1}(t)\big) \quad (i = 1, 2),$$
$$x_3'(t) = p_3(t)\, x_1\big(\tau_1(t)\big),$$

is considered, where $p_i \in L_{\mathrm{loc}}(R_+; R_+)$, $\tau_i \in C(R_+, R_+)$ and $\lim\limits_{t\to+\infty} \tau_i(t) = +\infty$ $(i = 1, 2, 3)$.

Sufficient conditions of new type are established for oscillation of solutions of the above system.

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1. Introduction

This work concerns the study of oscillatory properties of the system

$$\begin{aligned} &x_i'(t) = p_i(t)\, x_{i+1}\big(\tau_{i+1}(t)\big) \quad (i = 1, 2),\\ &x_3'(t) = p_3(t)\, x_1\big(\tau_1(t)\big), \end{aligned} \tag{1.1}$$

where

$$\begin{gathered} p_i \in L_{\mathrm{loc}}(R_+; R_+), \quad \tau_i \in C(R_+, R_+),\\ \lim_{t\to+\infty} \tau_i(t) = +\infty \quad (i = 1, 2, 3) \quad \text{and} \quad \tau_3'(t) \ge 0 \quad \text{for} \quad t \in R_+. \end{gathered} \tag{1.2}$$

The problem of oscillation of solutions of high order differential equations is well studied (see, for example, [1–4]). Analogous results for systems (1.1), where $p_3 \in L_{\mathrm{loc}}(R_+; R_-)$ is considered in [5].

Definition 1.1. Let $t_0 \in R_+$. A continuous vector function $\boldsymbol{x} = (x_i)_{i=1}^3 : [t_0; +\infty) \to R^3$ is said to be a proper solution of system (1.1) if it is locally continuous, $\sup\{\|x(s)\|.\, s \in [t_0; +\infty)\} > 0$ for $t \ge t_0$, there exists a function $\boldsymbol{x}^* \in C(R_+; R^3)$ such that $\boldsymbol{x}^* \equiv \boldsymbol{x}(t)$ on $[t_0, +\infty)$, and the equalities

$$x_i'(t) = p_i(t)\, x_{i+1}^*\big(\tau_{i+1}(t)\big) \quad (i = 1, 2),$$
$$x_3'(t) = p_3(t)\, x_1^*\big(\tau_1(t)\big),$$

hold for $t \ge t_0$.

Definition 1.2. A proper solution of the system (1.1) is said to be oscillatory if every component of at solution has a sequence of zeroes tending to $+\infty$. Otherwise the solution is said to be nonoscillatory.

2. Sufficient conditions for oscillation of solutions

Below, we will assume that the following conditions

$$\int_0^{+\infty} p_i(t)\,dt = +\infty \quad (i = 1, 2), \tag{2.1}$$

$$\int_0^{+\infty} p_3(t) \int_0^{\tau_1(t)} p_1(s) \int_0^{\tau_2(s)} p_2(\xi)\,d\xi\,ds\,dt = +\infty \tag{2.2}$$

and

$$\int_0^{+\infty} \tau_3'(t)\bigg(\int_0^t p_2(s)ds\bigg) h(\tau_3(t))\,h\big(\tau_1(\tau_3(t))\big)\,dt = +\infty \tag{2.3}$$

hold, where

$$h(t) = \int_0^t p_1(s)\,ds. \tag{2.4}$$

Theorem 2.1. *Let the conditions* (1.2), (2.1)–(2.3) *be fulfilled and for any* $\lambda \in [0, 1]$

$$\limsup_{\varepsilon\to 0+} \Bigg(\liminf_{t\to+\infty} (h(t))^{-\lambda-\delta_{2\varepsilon}(\lambda)} \int_0^t p_1(s)\big(h(\sigma(s))\big)^{\delta_{1\varepsilon}(\lambda)+\delta_{2\varepsilon}(\lambda)} \int_{\tau_2(s)}^{+\infty} \bigg(\int_{\tau_2(s)}^{\xi} p_2(\xi_1)d\xi_1\bigg)$$

$$\times\, p_3(\tau_3(\xi))\tau_3'(\xi)\big(h(\tau_1(\tau_3(\xi)))\big)^{\lambda-\delta_{1\varepsilon}} d\xi\,ds\Bigg) > 1.$$

Then each proper solution of the system (1.1) *either is oscillatory or satisfies the conditions*

$$\big|x_i(t)\big| \uparrow +\infty \quad (i = 1, 2, 3), \tag{2.5}$$

where

$$\sigma(t) = \overline{\sigma}(\overline{\tau}_2(t)), \quad \overline{\tau}_2(t) = \inf\big\{\min(s, \tau_2(s)) : s \ge t\big\},$$
$$\overline{\sigma}(t) = \inf\big\{\min(s, \tau_1(\tau_3(s))) : s \ge t\big\}, \tag{2.6}$$

$$\delta_{1\varepsilon}(\lambda) = \begin{cases} 0 & \text{for} \quad \lambda = 0, \\ \varepsilon & \text{for} \quad \lambda \in (0, 1], \end{cases} \qquad \delta_{2\varepsilon}(\lambda) = \begin{cases} 0 & \text{for} \quad \lambda = 1, \\ \varepsilon & \text{for} \quad \lambda \in [0, 1). \end{cases} \tag{2.7}$$

Theorem 2.2. *Let the conditions* (1.2), (2.1)–(2.3) *be fulfilled and*

$$\liminf_{t\to+\infty} \frac{h(\sigma(t))}{h(t)} > 0. \tag{2.8}$$

If, moreover, for any $\lambda \in [0,1]$

$$\limsup_{\varepsilon\to 0+}\left(\liminf_{t\to+\infty}\big(h(t)\big)^{-\lambda-\delta_{2\varepsilon}(\lambda)}\int_0^t p_1(s)\big(h(s)\big)^{\delta_{1\varepsilon}(\lambda)+\delta_{2\varepsilon}(\lambda)}\int_{\tau_2(s)}^{+\infty}\left(\int_{\tau_2(s)}^{\xi} p_2(\xi_1)d\xi_1\right)\right.$$
$$\left.\times\, p_3(\tau_3(\xi))\tau_3'(\xi)\big(h(\tau_1(\tau_3(\xi)))\big)^{\lambda-\delta_{1\varepsilon}(\lambda)}d\xi\, ds\right) > 1,$$

then each proper solution of the system (1.1) *either is oscillatory or satisfies the condition* (2.5)*, where the functions* $h,\sigma,\delta_{1\varepsilon},\delta_{2\varepsilon}$ *are defined by* (2.4)*,* (2.6) *and* (2.7)*.*

Theorem 2.3. *Let the conditions* (1.2)*,* (2.1)*–*(2.3)*,* (2.8) *be fulfilled and for any* $\lambda \in [0,1]$

$$\limsup_{\varepsilon\to 0+}\left(\liminf_{t\to+\infty}\big(h(t)\big)^{1-\lambda+\delta_{1\varepsilon}(\lambda)}\right.$$
$$\left.\times\int_{\tau_2(t)}^{+\infty}\left(\int_{\tau_2(t)}^{\xi} p_2(\xi_1)d\xi_1\right)p_3(\tau_3(\xi))\tau_3'(\xi)\big(h(\tau_1(\tau_3(\xi)))\big)^{\lambda-\delta_{1\varepsilon}(\lambda)}d\xi\right) > \lambda.$$

Then each proper solution of the system (1.1) *either is oscillatory or satisfies the condition* (2.5)*, where the functions* $h,\delta_{1\varepsilon}$ *are defined by* (2.4) *and* (2.7)*.*

Theorem 2.4. *Let the conditions* (1.2)*,* (2.1)*–*(2.3)*,* (2.8) *be fulfilled and there exist* $c_*>0$ *and* $\tau_*>0$ *such that*

$$p_2(t)\ge c_*p_1(t)\quad \textit{for}\quad t\in R_+\quad \textit{and}\quad \liminf_{t\to+\infty}\frac{h(\tau_2(t))}{h(t)}=\tau_*. \tag{2.9}$$

If, moreover for any $\lambda \in [0,1]$

$$\limsup_{\varepsilon\to 0+}\left(\liminf_{t\to+\infty}\big(h(t)\big)^{1-\lambda+\delta_{1\varepsilon}(\lambda)}\int_t^{+\infty} h(s)p_3(\tau_3(s))\tau_3'(s)\big(h(\tau_1(\tau_3(s)))\big)^{\lambda-\delta_{1\varepsilon}(\lambda)}ds\right)$$
$$>\frac{\lambda(2-\lambda)}{c_*\,\tau_*^{\lambda}},$$

then each proper solution of the system (1.1) *either is oscillatory or satisfies the condition* (2.5)*.*

Theorem 2.5. *Let the conditions* (1.2)*,* (2.1)*–*(2.3)*,* (2.8)*,* (2.9) *be fulfilled and for any* $\lambda \in [0,1]$

$$\limsup_{\varepsilon\to 0+}\left(\liminf_{t\to+\infty}h(t)\int_t^{+\infty}\big(h(s)\big)^{1-\lambda+\delta_{1\varepsilon}(\lambda)}p_3(\tau_3(s))\tau_3'(s)\big(h(\tau_1(\tau_3(s)))\big)^{\lambda-\delta_{1\varepsilon}(\lambda)}ds\right)$$
$$>\frac{\lambda(1-\lambda)(2-\lambda)}{c_*\,\tau_*^{\lambda}}\,.$$

Then each proper solution of the system (1.1) *either is oscillatory or satisfies the condition* (2.5).

Theorem 2.6. *Let* $\alpha_i, c_i \in (0, +\infty)$ $(i = 1, 2, 3)$. *Then for every proper solution of the system*

$$x_i'(t) = c_i\, x_{i+1}(\alpha_{i+1}\, t) \quad (i = 1, 2),$$
$$x_3'(t) = \frac{c_3}{t^3}\, x_1(\alpha_1\, t),$$

either is oscillatory or satisfies the condition (2.5), *if only if the inequality*

$$c_1\, c_2\, c_3\, \alpha_1^{-1} \alpha_3^{-1} > \max\left\{\lambda(1-\lambda)(2-\lambda)(\alpha_1\alpha_2\alpha_3)^{-\lambda} : \lambda \in [0, 1]\right\}$$

holds.

R E F E R E N C E S

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