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THE SECOND BVP OF THE THEORY OF ELASTIC BINARY MIXTURES FOR A PLANE WITH CURVILINEAR CUTS

Bitsadze L.

Abstract. The second boundary value problems of the theory of elastic binary mixtures for a transversally isotropic plane with curvilinear cuts is investigated. The solvability of a system of singular integral equations is proved by using the potential method and the theory of singular integral equations.

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Introduction

In the present paper the second boundary value problem (BVP) of elastic binary mixture theory is investigated for a transversally-isotropic plane with curvilinear cuts. The boundary value problems of elasticity for anisotropic media with cuts were considered in [1,2]. In this paper we intend this result to BVP for a transversally-isotropic elastic mixture. Using the potential method and the theory of singular integral equations we prove the solvability of system of singular integral equations, corresponding to the boundary value problem. One more particular case is discussed when a rectilinear cut is given on Ox_1 axis.

The basic homogeneous equations of statics of the transversally isotropic elastic binary mixtures theory in the case of plane deformation can be written in the form [3]

$$C(\partial x)U = \begin{pmatrix} C^{(1)}(\partial x) & C^{(3)}(\partial x) \\ C^{(3)}(\partial x) & C^{(2)}(\partial x) \end{pmatrix} U = 0, \tag{1}$$

where the components of the matrix $C^{(j)}(\partial x) = \|C^{(j)}_{pq}(\partial x)\|_{2x2}$ are given in the form

$$C^{(j)}_{11}(\partial x) = c^{(j)}_{11}\frac{\partial^2}{\partial x_1^2} + c^{(j)}_{44}\frac{\partial^2}{\partial x_3^2}, \quad C^{(j)}_{12}(\partial x) = (c^{(j)}_{13} + c^{(j)}_{44})\frac{\partial^2}{\partial x_1 \partial x_3},$$
$$C^{(j)}_{22}(\partial x) = c^{(j)}_{44}\frac{\partial^2}{\partial x_1^2} + c^{(j)}_{33}\frac{\partial^2}{\partial x_3^2}, \quad C^{(j)}_{pq}(\partial x) = C^{(j)}_{qp}(\partial x), j = 1, 2, 3; p, q = 1, 2,$$

$c^{(k)}_{pq}$ are constants, $U = U(x) = (u', u'')$ is four-dimensional displacement column vector-function, $u'(x) = (u'_1, u'_3)$ and $u''(x) = (u''_1, u''_3)$ are partial displacement vectors depending on the variables x_1, x_3.

Let the plane be weakened by curvilinear cuts $l_j = a_j b_j, j = 1, 2, ..., p$. Assume that l_j are simple relatively nonintersecting open Lyapunov arcs. The direction from a_j to b_j is taken as the positive one on l_j. The normal to l_j will be drawn to the right relative to motion in the positive direction. Let's denote by D the infinite plane with curvilinear cuts $l_j = a_j b_j,\ \ j=1,2,...,p,\ l=\bigcup_{j=1}^{p} l_j$.

We introduce the notations: $z = x_1 + ix_3,\quad \zeta_k = y_1 + \alpha_k y_3,\quad t_k = t_1 + \alpha_k t_3,$ $z_k = x_1 + \alpha_k x_3,\ \ t = t_1 + it_3$.

The second boundary value problem of static of the theory of elastic binary mixtures is formulated as follows:

Problem 2. Find a regular solution of the equation (1) in D, when the stress vector is given on both sides of the $l_j,\ \ j=1,2,...,p$. In addition, it is assumed that the principal vector of external force acting on l, stress vector and the rotation at infinity are zero. The boundary conditions can be written as follows:

$$[TU]^+(t_0) = f^+(t_0),\quad [TU]^-(t_0) = f^-(t_0),\quad t_0 \in l, \tag{2}$$

where f^+ and f^- are the given known vector-functions satisfying Holder's condition on the arcs l_j, having the derivatives in the class H^* (for the definitions of the classes H and H^* see [4]) and satisfying the following conditions at the endpoints a_j and b_j of the arcs l_j

$$f^+(a_j) = f^-(a_j),\quad f^+(b_j) = f^-(b_j),\ \ j=1,2,..,p.$$

Here symbol (+) corresponds to the boundary value on the left-hand side of l_j for motion from the point a_j to the point b_j.

The stress vector is defined as follows ([5], [6])

$$T(\partial x, n)U = \begin{pmatrix} T^{(1)}(\partial x, n) & T^{(3)}(\partial x, n) \\ T^{(3)}(\partial x, n) & T^{(2)}(\partial x, n) \end{pmatrix} U, \tag{3}$$

where

$$T^j(\partial x, n) = \begin{pmatrix} c_{11}^{(j)} n_1 \partial x_1 + c_{44}^{(j)} n_3 \partial x_3 & c_{13}^{(j)} n_1 \partial x_3 + c_{44}^{(j)} n_3 \partial x_1 \\ c_{44}^{(j)} n_1 \partial x_3 + c_{13}^{(j)} n_3 \partial x_1 & c_{44}^{(j)} n_1 \partial x_1 + c_{33}^{(j)} n_3 \partial x_3 \end{pmatrix}, j=1,2,3, \tag{4}$$

Here n_1, n_3 are the components of normal vector, at the point $y \in l_j$, $\partial x_k = \dfrac{\partial}{\partial x_k}$.

We seek the solution of the above formulated problem in the form [5],

$$U(z) = \frac{1}{\pi} Im \sum_{k=1}^{4} R_{(k)}^T iL \int_l ln(z_k - \zeta_k)[g(t) + ih(t)]ds,$$

where g and h are unknown real vector-functions from the Holder class, $R_{pq}^{(k)}, L$ are given in [5-6].

For the stress vector we get

$$T(\partial x, n)U(z) = \frac{1}{\pi} Re \sum_{k=1}^{4} P_{(k)} \int_l \frac{\partial ln(z_k - \zeta_k)}{\partial s}[g(t) + ih(t)]ds, \tag{5}$$

where $P_{(k)} = L^{(k)}L$,

$$L^{(k)} = \begin{pmatrix} \alpha_k^2 L_{22}^{(k)} & -\alpha_k L_{22}^{(k)} & \alpha_k^2 L_{24}^{(k)} & -\alpha_k L_{24}^{(k)} \\ -\alpha_k L_{22}^{(k)} & L_{22}^{(2)} & -\alpha_k L_{24}^{(k)} & L_{24}^{(k)} \\ \alpha_k^2 L_{24}^{(k)} & -\alpha_k L_{24}^{(k)} & \alpha_k^2 L_{44}^{(k)} & -\alpha_k L_{44}^{(k)} \\ -\alpha_k L_{24}^{(k)} & L_{24}^{(k)} & -\alpha_k L_{44}^{(k)} & L_{44}^{(k)} \end{pmatrix},$$

$$L_{22}^{(k)} = -\Delta q_4 d_k [a_{44} + \alpha_k^2(b_{11} + 2a_{34}) + a_{33}\alpha_k^4],$$

$$L_{24}^{(k)} = \Delta q_4 d_k [a_{24} + \alpha_k^2(-b_{33} + a_{14} + a_{23}) + a_{13}\alpha_k^4],$$

$$L_{44}^{(k)} = -\Delta q_4 d_k [a_{22} + \alpha_k^2(b_{22} + 2a_{12}) + a_{11}\alpha_k^4],$$

$a_{ij}, \Delta, d_k, q_k, b_{ij}$, are given in [5-6].

To define the unknown density, by virtue of (2)-(5), we obtain the following system of singular integral equations of normal type

$$\pm g(t) + \frac{1}{\pi} Re \sum_{k=1}^{4} P_{(k)} \int_l \frac{\partial ln(t_{k0} - t_k)}{\partial s} [g(t) + ih(t)] ds = f^{\pm}. \tag{6}$$

From (6) we deduce that

$$\begin{aligned} &2g(t) = f^+(t_0) - f^-(t_0), \\ &\frac{1}{\pi} Re \sum_{k=1}^{4} P_{(k)} \int_l \frac{\partial ln(t_{k0} - t_k)}{\partial s} h(t) ds = \Omega(t_0), t_0 \in l, \end{aligned} \tag{7}$$

where $\Omega(t_0)$ is given by

$$\Omega(t_0) = \frac{1}{2}(f^+(t_0) + f^-(t_0)) - \frac{1}{\pi} Re \sum_{k=1}^{4} P_{(k)} \int_l \frac{\partial ln(t_{k0} - t_k)}{\partial s} [f^+(t) - f^-(t)] ds,$$

Thus, we have defined the vector g on l. It is not difficult to verify, that $g \in H, g' \in H^*$, and $\Omega \in H, \Omega' \in H^*$. (7) is a system of singular integral equations of normal type with respect to the vector h. We seek the solution of the system (7) in the class h_0 (for the definition of the class h_0 see [4]). Points a_j and b_j are nonsingular ones and the total index in the class h_0 is equal to $-3p$. Let's prove that the adjoint homogeneous equation corresponding to the system (7) has only the trivial solution in the adjoint class.

The adjoint homogeneous system of singular integral equations has the form

$$\frac{1}{\pi} Re \sum_{k=1}^{4} LL^{(k)} \int_l \frac{\partial ln(t_k - t_{k0})}{\partial s} \nu(t) ds = 0. \tag{8}$$

If the solution of equation (8) in the adjoint class exist, it will satisfy the Holder's condition on l, vanishing at the points a_j and $b_j, j = 1, 2, ..p$, and having the derivatives in the class H^* ([4]).

Multiplying the system (8) by matrix $a = L^{-1}$, and taking into account the identity $aLL^{(k)} = P_{(k)}a$, we obtain

$$\frac{1}{\pi} Re \sum_{k=1}^{4} P_{(k)} \int_l \frac{\partial ln(t_k - t_{k0})}{\partial s} a\nu(t) ds = 0. \tag{9}$$

Let's assume that (9) has nontrivial solution ν_0 in the adjoint class and construct the potential

$$u_0(z) = \frac{1}{\pi} Re \sum_{k=1}^{4} R^{(k)T} L \int_l \frac{\partial ln(t_k - z_k)}{\partial s} a\nu_0(t) ds. \tag{10}$$

From (10) we obtain

$$T(\partial z, n) u_0 = \frac{\partial \Phi(z)}{\partial s},$$

where

$$\Phi(z) = \frac{1}{\pi} Re \sum_{k=1}^{4} P_{(k)} \int_l \frac{\partial ln(t_k - z_k)}{\partial s} a\nu_0(t) ds.$$

By virtue of (9) it is obvious that $\Phi^{\pm}(t_0) = 0, t_0 \in l$. On the basis of the uniqueness theorem we conclude,that $u(t_0) = 0$. Then from equality $u_0^+ - u_0^- = 2const\nu_0$, it follows that $\nu_0 = 0, t \in l$. Consequently, it follows that the systems (8) and (9) have only the trivial solution.

Thus the homogeneous system corresponding to the system (9) has only $4p$ linearly independent solution.Threrefore, the corresponding nonhomogeneous system is solvable in the adjoint class and the solution depends on the 4p arbitrary constants $K_1, K_2, .., K_{4p}$. The choice of these constants stipulates by conditions follows from the single-valuedness of the displacement vector. The displacement vector obtains the increment, while going around l_j, that has to vanish

$$\int_l h(t) ds = RL \int_l (f^+ - f^-) ds, \tag{11}$$

where $\sum_{k=1}^{4} R^{(k)T} = -E + iR$.

Hence (11) is an algebraic equation with respect to unknown constants K_j. Let's prove that the determinant of this system is not zero. In fact, let's take the homogeneous system, corresponding to the conditions $f^+ = 0, f^- = 0, (TU)^\infty = 0$. Supposing the solution $K_j^{(0)}, j = 1, \ldots, 4p$, to be nontrivial, we construct the potential

$$U_0(z) = \frac{1}{\pi} Re \sum_{k=1}^{4} R^{(k)T} L \int_l ln(t_k - z_k) h_0(t) ds, \tag{12}$$

where h_0 is a linear combination of solution $h^{(j)}$. $h_0 = \sum\limits_{k=1}^{4p} K_j^{(0)} h^{(j)}$, and $h^{(j)}$ are linearly independent solutions of the homogeneous equation corresponding to (7). $h^{(j)}$ have to satisfy the following condition $\int\limits_{l_j} h^{(0)} ds = 0, j = 1, \ldots, p.$

Then the potential (12) is regular at infinity and by the uniqueness theorem $u_0 = 0$. But we have the following equality

$$\left(\frac{\partial u_0}{\partial s}\right)^+ - \left(\frac{\partial u_0}{\partial s}\right)^- = Lh^{(0)} = 0.$$

Hence we conclude that $K_j = 0$, which contradicts the assumption. Thus the solvability of the problem is proved.

Let us consider a particular case, when the plane has only one rectilinear cut ab along the real axis. Assuming that the principal vector of external forces vanishes at infinity. Then the stress vector outside of the ab is calculated by the formula

$$TU_0(z) = \frac{1}{2\pi} Re \sum_{k=1}^{4} \left[P_{(k)} \int\limits_l \frac{f^+ - f^-}{t - z_k} ds + \frac{1}{X(z_k)} \int\limits_l \frac{X^+(t)(f^+ + f^-)}{t - z_k} dt \right],$$

where $X(z_k) = \sqrt{z_k - a)(b - z_k)}$ is a holomorphic function on the plane cut along the arc ab and $X^+(t) = \sqrt{t - a)(b - t)}$.

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Author's address:

L. Bitsadze
I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: lamarabits@yahoo.com