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COMPLEX POINTS OF RANDOM SURFACES

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Abstract. We deal with complex points of two-dimensional surfaces. A short review of basic results about complex points of smooth surfaces in $\mathbb{C}^2$ is presented at the beginning. Some estimates for the expected number of complex points of a random planar endomorphism are obtained.

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Complex points of smooth two-dimensional surfaces in $\mathbb{C}^2$ play significant role in some problems of complex analysis, and symplectic geometry. For example, existence of complex points sometimes enables one to construct so-called *attached analytic discs* with boundaries in a given surface. Correspondingly, complex points are related to the so-called *Bishop problem* about the existence of analytic discs attached to the graph of an arbitrary smooth complex-valued function in the plane. Despite considerable progress, this problem is not yet completely solved and complex points of two-dimensional surfaces attract permanent interest, [8].

There are also some open problems concerned with complex points on compact surfaces. It is remarkable that in the setting of compact surfaces this problem has interesting topological aspects. In particular, it is well-known that the geometry of complex points on a compact surface X is closely related to its Euler characteristic $\chi(X)$.

We start by recalling necessary information about complex points and random Gaussian surfaces. Next, we describe two examples of random polynomials appropriate for our setting. Finally, we present the main result which is concerned with the asymptotic of the expected number of complex points.

Definition 1. Let X be a smooth oriented two-dimensional surface in $\mathbb{C}^2$. A point $p \in X$ is called a complex point of X if the tangent plane T_pX is a complex line in $\mathbb{C}^2$.

As was shown in recent papers [10], [3], [4], [7], [9], [6], computing topological invariants of random polynomials leads to interesting problems and results. In our context, a natural problem in spirit of the named approach is to compute or estimate the expected number of complex points of a Gaussian *random planar endomorphism* which we call *random plend.*

It is well known that a Gaussian random planar endomorphism of fixed algebraic degree is almost surely (i.e., with probability one) proper and stable in the sense of Whitney [9], [6] which makes it possible to introduce several useful numerical characteristics of its geometric behaviour such as the topological degree and the number of cusps. Hence one can consider the expected values of these characteristics and obtain in this way numerical invariants of a given random planar endomorphism. This strategy was used in [7], [6], [2] and led to a number of new results.

Following the same strategy, it is easy to show that a random plend of fixed algebraic degree almost surely has a finite number of complex points all of which are transversal. Thus it makes sense to speak of the *expected number* of complex points of a random plend. We study this invariant using the techniques described in [5] and then briefly outline how it can be related to the two other invariants.

Recall that the *expected value of the topological degree* was found in [6] for random planar endomorphism defined by the gradient of random polynomial with a certain rotation invariant Gaussian distribution of coefficients introduced in [10]. Since Maslov index $M(F)$ [8] of a plend F can be expressed as the topological degree of associated plend $\overline{\partial}F$ components of which are linear combination of partial derivatives of components of F, it turns out that estimates for the expected gradient degree appear applicable for Maslov index. If all coefficients of random polynomial are standard normals, we find the asymptotic of the expected value of $C(F)$ as the algebraic degree of the random plend tends to infinity.

As was already mentioned, we deal with *Gaussian random polynomials*, which means that their coefficients are real random variables and have multivariate normal distribution. The term "random polynomial" always refers to this situation. To be more precise, we reproduce some concepts and notation from probability theory. All necessary background can be found in [3].

Recall that a (one-dimensional) Gaussian (normal) random variable ξ is defined as a real-valued random variable with Gaussian (normal) density

$$f_\xi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}},$$

where $\sigma > 0$, $-\infty < a < +\infty$. Parameters a and σ determine its expectation and variance:

$$a = E\xi,\ \sigma^2 = D\xi.$$

If $a = 0$, we speak of a *central normal distribution.* If $a = 0$ and $\sigma = 1$ one obtains the *standard normal distribution*

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du.$$

Recall also that, for a pair of random variables (ξ, η), the number

$$\operatorname{cov}(\xi, \eta) = E\big[(\xi - E\xi) \cdot (\eta - E\eta)\big]$$

is called *covariation* of ξ and η. If $\operatorname{cov}(\xi, \eta) = 0$, then ξ and η are called non-correlated, in particular (stochastically) independent random variables are non-correlated. The *variance* $D\xi$ is defined as $\operatorname{cov}(\xi, \xi) = D\xi$.

Recall finally that a *multidimensional central normal distribution* Ξ is defined as a distribution with density

$$f_\Xi(x) = \frac{\sqrt{|A|}}{(2\pi)^{\frac{n}{2}}} \cdot e^{-\frac{1}{2}\cdot Q(x)},$$

where

$$Q(x) = xAx^T = \sum_{i,j=1}^{n} a_{ij}x_ix_j,$$

and $|A|$ denotes the determinant of a positive definite matrix $A = (a_{ij})$. The word *central* indicates again that $E\Xi = 0$ (sometimes one says that this is a distribution with *zero mean* [3]). Matrix A is called the *covariation matrix* of the above distribution because its elements are given by pairwise covariations of the components of distribution Ξ.

Let P be a Gaussian random polynomial on $\mathbb{R}^n$ in the above sense. Then it can be written as

$$P(x) = \sum_{\alpha} \xi_\alpha x^\alpha$$

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $x = (x_1, x_2, \dots, x_n)$, $x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \dots \cdot x_n^{\alpha_n}$, where ξ_α have multidimensional normal distribution. To specify this distribution it is sufficient to give its covariation matrix explicitly. Denote by m the algebraic degree of P.

Example 1. Let all coefficients ξ_α be i.i.d. (independent identically distributed) standard normals. Then P is called a standard Gaussian random polynomial in n variables of algebraic degree m. Taking n copies of independent standard Gaussian polynomials we get standard random endomorphism in $\mathbb{R}^n$.

It turned out that in the case of several variables such polynomials are sufficiently difficult to work with. We also give the following example of Gaussian random polynomial introduced in [10].

Example 2. Denote by $H_m(\mathbb{R}^{n+1})$ the set of all homogeneous polynomials of algebraic degree m (m-forms) on $\mathbb{R}^{n+1}$ and consider a Gaussian random homogeneous polynomial f from $H_m(\mathbb{R}^{n+1})$ having the form

$$F(x) = \sum_{m_0+m_1+\ldots+m_n=m} F_{m_0,m_1,\ldots,m_n} x_0^{m_0} x_1^{m_1} \ldots x_n^{m_n}, \ x \in \mathbb{R}^{n+1},$$

where $F_{m_0,\ldots,m_n}$ are independent normal random variables with zero mean and variances

$$EF^2_{m_0,\ldots,m_n} = \frac{m!}{m_o! \ldots m_n!}.$$

We call it a convenient Gaussian random m-form.

It can be verified that such a random polynomial is invariant with respect to the natural action of the group $O(n+1)$ on $H_m(\mathbb{R}^{n+1})$ for which reason it is said to be *rotation invariant*. As was shown in [9], an important characteristic of a rotation invariant Gaussian random polynomial is given by the so-called *parameter* r which is defined by the formula

$$r = \frac{E\left(\frac{\partial F}{\partial x_0}(e)\right)^2}{EF(e)^2},$$

where $e = (1, 0, \dots, 0) \in S^n$ is a chosen point. It is possible to check by a direct computation that in the above example we get $r = m$. Taking n independent random polynomials of such type we come to the concept of *convenient random endomorphism*.

There is one more type of random polynomial which is relevant for our purposes. It was introduced in [9] and can be described as follows. Introduce a scalar product on $H_m(\mathbb{R}^{n+1})$ by the formula

$$(f, f') = \int_{x \in S^n} f(x) f'(x) dx, \quad f, f' \in H_m(\mathbb{R}^{n+1}).$$

Gaussian random m-form F is called satisfactory if

$$E\big((f, F)(f', F)\big) = (f, f'), \quad f, f' \in H_m(\mathbb{R}^{n+1}).$$

According to [9] such a random polynomial is rotation invariant and its parameter r is equal to

$$\frac{m(m+n+1)}{n+2}.$$

Consider now a general (i.e. not necessarily homogeneous) Gaussian random polynomial on $\mathbb{R}^{n+1}$ of algebraic degree N such that each of its homogeneous components of degree $m \leq N$ has *convenient* Gaussian distribution specified in Example 2. In such situation we speak of a *convenient random polynomial* of algebraic degree N. This is precisely the type of random polynomial we wish to consider. There are (at least) two main reasons for such choice: first, such polynomials and their numerical invariants appeared useful in physics, and, second, for systems of random polynomials of such type M. Shub and S. Smale [10] obtained simple explicit formulae for the expected number of real roots, which enabled one to obtain a lot of results about numerical invariants of such polynomials [3].

One can analogously define a (non-homogeneous) *satisfactory random polynomial* of algebraic degree N. For such polynomials, one can also obtain some useful results about their numerical invariants.

After these preparations we are ready to deal with the complex points of Gaussian random plends. First, notice that in this case components of $\overline{\partial F}$ are Gaussian random polynomials. This follows from explicit formulae for the coefficients of $\overline{\partial F}$ and from the fact that sum of independent Gaussian variables is again a Gaussian random variable.

It is easy to check that, for convenient random plend, components of $\overline{\partial F}$ are rotation invariant. Moreover, by direct verification one can check that the parameters of components of convenient plend are both equal to the parameters of the gradient of convenient random polynomial. Since the expected value of rotation invariant Gaussian plend is completely determined by its parameter, we can use the main result of [6], which directly leads to the desired result.

Theorem 1. *The expected number of complex points $E(C(F))$ of a convenient random plend F of algebraic degree m is asymptotically equivalent to $\sqrt{m}$ as $m \to \infty$.*

Similar results can be obtained for some other types of random surfaces.

REFERENCES

1. Aliashvili T. Counting real roots of polynomial endomorphisms. *J. Math. Sci.* **118**, 5 (2003), 5325-5346.

2. Aliashvili T. On invariants of random planar endomorphisms, *Banach Center Publ.* **62** (2004), 19-28.

3. Edelman A., Kostlan E. How many roots of a real polynomial are real? *Bull. Amer. Math. Soc.* **32** (1995), 1-37.

4. Ibragimov I., Podkorytov S. On random algebraic surfaces. (Russian) *Dokl. Ross. Akad. Nauk* **343** (1995) 734-736.

5. Khimshiashvili G. Signature formulae for topological invariants. *Proc. A.Razmadze Math. Inst.* **125** (2001), 1-121.

6. Khimshiashvili G. Topological aspects of random polynomials. *Bull. Georgian Acad. Sci.* **168** (2003), 5-8.

7. Khimshiashvili G., Ushveridze A. On the average topological degree of random polynomials. *Bull. Georgian Acad. Sci.* **159** (1999), 385-388.

8. Khimshiashvili G., Wegert E. Complex points of planar endomorphisms. *Bull. Georgian Acad. Sci.* **165** (2002), 26-29.

9. Podkorytov S. The mean value of Euler characteristic of random algebraic hypersurface. (Russian) *Algebra i Analiz,* **11** (1999), 185-193.

10. Shub M., Smale S. Complexity of Bezout's theorem. II: Volumes and probabilities. *Progr. Math.*, **109** (1993), 267-285.

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