

THE CAUCHY-LEBESQUE TYPE INTEGRALS FOR THE GENERALIZED
BELTRAMI SYSTEMS

Akhalaia G., Manjavidze N.

Abstract. The generalized Beltrami systems in the complex plane are considered. The generalized Cauchy-Lebesgue type integrals for these systems are introduced and some properties of these integrals are established.

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Consider the first order system of partial differential equations in the complex plane C

$$w_{\bar{z}} = Q w_z, \quad (1)$$

where Q is given $n \times n$ complex matrix of the class $W_p^1(C)$, $p > 2$ and $Q(z) = 0$ outside of some circle. (The notation $A \in K$, where A is a matrix and K is some class of functions, means that every element $A_{\alpha\beta}$ of A belongs to K . If K is some linear normed space with the norm $\|\cdot\|_K$, then $\|A\|_K = \max_{\alpha,\beta} \|A_{\alpha\beta}\|_K$.)

Bojarski [1] assumed that the variable matrix Q in (1) is lower quasideagonal matrix of the special form having the eigenvalues less than 1. Hile [2] noticed that what appears to be the essential property of the elliptic systems of the form (1) for which one can obtain a useful extension of the analytic function theory is the self-commuting property of the matrix Q , which is

$$Q(z_1) Q(z_2) = Q(z_2) Q(z_1) \quad (2)$$

for any two points $z_1, z_2 \in C$. Following Hile if Q is self-commuting in C , and if $Q(z)$ has eigenvalues the less than 1, then the system (1) is called generalized Beltrami system. Solutions of such a system will be called Q -holomorphic vectors. Under the solution of the equation (1) in some domain D we understand so-called regular solution (see [4]) i.e. $w(z) \in L_2(\bar{D})$, whose generalized derivatives $w_{\bar{z}}, w_z$ belong to $L_r(D')$, $r > 2$, where $D' \subset D$ is an arbitrary closed subset. Equation (1) is to be satisfied almost everywhere in D .

Remark. We say that A and B commute in D if A and B are matrix valued functions defined in D and satisfy the condition

$$A(z_1) B(z_2) = B(z_2) A(z_1) \quad \text{for all } z_1, z_2 \in D.$$

It is well-known, that if $\{Q_\alpha\}$ is a commutative collection of complex $n \times n$ matrices then there exists an invertible complex $n \times n$ matrix S such that for each Q_α

$$S Q_\alpha S^{-1} = \begin{bmatrix} A_{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & A_{\alpha_m} \end{bmatrix}$$

and each A_{α_j} has the following form:

$$A_{\alpha_j} = \begin{bmatrix} \lambda_{\alpha_j} & & 0 \\ & \ddots & \\ * & & \lambda_{\alpha_j} \end{bmatrix}.$$

Each square block matrix A_{α_j} has the same size for all α , but may change with respect to j (the star-denotes possibly nonzero terms). This shows, in particular, that the self-commuting condition is really very strong condition.

The matrix valued function $\Phi(z)$ is a generating solution of the system (1) if it satisfies the following properties ([2]):

- (i) $\Phi(z)$ is a C^1 -solution of (1) in C ;
- (ii) $\Phi(z)$ is self-commuting and commutes with Q in C ;
- (iii) $\Phi(t) - \Phi(z)$ is invertible for all z, t in C , $z \neq t$;
- (iv) $\partial_{\bar{z}}\Phi(z)$ is invertible for all z in C .

We call the matrix

$$V(t, z) = \partial_t \Phi(t) [\Phi(t) - \Phi(z)]^{-1}$$

the generalized Cauchy kernel for the system (1).

Let now Γ be a union of simple closed non-intersecting Liapunov-smooth curves, bounding finite or infinite domain. If Γ is one closed curve then D^+ denotes the finite domain; if Γ consists of several curves then by D^+ we denote the connected domain the boundary of which is Γ . On these curves the positive direction is chosen such, that when moving to this direction the domain D^+ remains left; the complement of open set $D^+ \cup \Gamma$ in the whole plane denote by D^- .

Assume the vector $\varphi(t) \in L(\Gamma)$ is given and consider the following integral:

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} V(t, z) d_Q t \varphi(t), \tag{3}$$

where $d_Q t = I dt + Q d\bar{t}$, I is a identity matrix. It is evident, that $\Phi(z)$ is Q -holomorphic vector everywhere outside of Γ , $\Phi(\infty) = 0$. We call the vector $\Phi(z)$ the generalized Cauchy-Lebesgue type integral for the system (1) with the jump line Γ . The boundary values of $\Phi(z)$ on Γ are given by the formulas:

$$\Phi^{\pm} = \pm \frac{1}{2} \varphi(t) + \frac{1}{2\pi i} \int_{\Gamma} V(\tau, t) d_Q \tau \mu(\tau). \tag{4}$$

The formulas (4) are to be fulfilled almost everywhere on Γ , provided that Φ^{\pm} are angular boundary values of the vector $\Phi(z)$ and the integral in (4) is to be understood in the sense of Cauchy principal value.

Theorem 1. *Let $\Phi(z)$ be a Q -holomorphic vector on the plane cut along the Γ , $\Phi(\infty) = 0$. Let $\Phi(z)$ have the finite angular boundary values Φ^+ and Φ^- inside and outside of Γ respectively. For the vector $\Phi(z)$ to be representable by the Cauchy-Lebesgue type integral (3) with the jump line Γ , it is necessary and sufficient that the equality*

$$\frac{1}{\pi i} \int_{\Gamma} V(t, t_0) d_Q t [\Phi^+(t) - \Phi^-(t)] = \Phi^+(t_0) + \Phi^-(t_0) \tag{5}$$

be fulfilled almost everywhere on Γ (cf. [4]).

(The equality (5) includes the condition $\Phi^+(t) - \Phi^-(t) \in L(\Gamma)$).

We call the generalized Cauchy-Lebesgue type integral (3) the generalized Cauchy-Lebesgue integral in the domain D^+ (in D^-) if $\Phi^+(t) = \varphi(t)$ ($\Phi^-(t) = -\varphi(t)$) almost everywhere on Γ .

From the Theorem 1 it is easy to conclude the following.

Theorem 2. *Let $\Phi(z)$ be a Q -holomorphic vector in D^+ and $\Phi^+(t) \in L(\Gamma)$ (in D^- , $\Phi(\infty) = 0$ and $\Phi^-(t) \in L(\Gamma)$). For this vector to be representable by the generalized Cauchy-Lebesgue integral in D^+ (in D^-) it is necessary and sufficient that the equality*

$$\begin{aligned} \frac{1}{\pi i} \int_{\Gamma} V(t, t_0) d_Q t \Phi^+(t) &= \Phi^+(t_0) \\ \left(\frac{1}{\pi i} \int_{\Gamma} V(t, t_0) d_Q t \Phi^-(t) &= -\Phi^-(t_0) \right) \end{aligned}$$

be fulfilled almost everywhere on Γ .

Theorem 3. *Let $\Phi(z)$ be a Q -holomorphic vector representable by the generalized Cauchy-Lebesgue type integral in D^+ and let $\Phi^+(t) \in L(\Gamma)$. Then $\Phi(z)$ is representable by the generalized Cauchy-Lebesgue integral with respect to its boundary values. The analogous conclusion for the infinite domain D^- in case $\Phi(\infty) = 0$ is also valid.*

Introduce now some classes of Q -holomorphic vectors. We say that Q -holomorphic vector $\Phi(z)$ belongs to the class $E_p(D^+, Q)$ [$(E_p(D^-, Q))$], $p > 1$, if $\Phi(z)$ is representable by the generalized Cauchy-Lebesgue type integral with the density from the class $L_p(\Gamma)$. It is easy to deduce from the Theorem 3 that every Q -holomorphic vector from the class $E_p(D^\pm, Q)$ is representable by the generalized Cauchy-Lebesgue integral with respect to its angular boundary values.

The following theorem is very important for the investigation of the Riemann-Hilbert type problems. One can prove it almost repeating the arguments of Bojarski [1] (see also [3]).

Theorem 4. *Let D be a domain of the complex plane bounded by the union of simple closed Liapunov-smooth curves $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \dots \cup \Gamma_m$ without common points, $\Gamma_1, \dots, \Gamma_m$ are situated outside of each other, but inside of Γ_0 (Γ_0 may be missing, then D is infinite domain). Every Q -holomorphic vector of the class $E_p(D, Q)$ admits the following representation*

$$\Phi(z) = \frac{1}{\pi i} \int_{\Gamma} V(t, s) d_Q t \mu(t) + iC, \quad (6)$$

where $\mu(t) \in L_p(\Gamma)$ is real vector, C is real constant vector. The vector $\mu(t)$ is defined uniquely to within the constant vector on Γ_j , $j \geq 1$, $\mu(t)$ on Γ_0 and C are defined uniquely by the vector $\Phi(z)$.

R E F E R E N C E S

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Authors' addresses:

G. Akhalaia

I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: giaakha@gmail.com

N. Manjavidze

Georgian Technical University
77, Kostava St., Tbilisi 0175
Georgia
E-mail: Ninomanjavidze@yahoo.com