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ON THE CONSTRUCTION OF SOLUTIONS OF THE SPATIAL AXIALLY SYMMETRIC STATIONARY PROBLEMS WITH PARTIALLY UNKNOWN BOUNDARIES OF THE THEORY OF JET FLOWS

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Abstract. We present a general mathematical method of constructing stationary solutions of spatial axially symmetric problems, with partially unknown boundaries, of jet theory, in particular, we consider a liquid flow of finite width round a spatial circular wedge. Unknown functions (velocity potential, flow function) and their arguments on each interval of the boundary satisfy two inhomogeneous boundary conditions. The system of differential equations with respect to the velocity potential and flow function are reduced to the normal form. Unknown functions are represented by a sum of holomorphic and generalized analytic functions. One problem from the theory of spatial jets is solved.

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In the present work we present a general mathematical method of solutions of spatial axially symmetric with partially unknown boundaries problems of the theory of jet flows, in particular, we consider spatial axially symmetric jet flows ([1, 2]).

The symmetry axis coincides with the x-axis, and the distance to the x-axis is denoted by y. The use is made of the right coordinate system.

Of an infinite set of half-planes passing through the x-axis, we choose arbitrarily one. But for the effectiveness, it is sometimes better to take two symmetric half-planes lying in one plane. The boundary of the domain under consideration consists of known and unknown parts. The known parts of the boundary are represented by the straight lines and their segments, while unknown portions of the boundary consist of two curves. Every segment of the boundary has two boundary conditions. The unknown functions (velocity potential, flow function) and their arguments on every segment of the boundary satisfy two inhomogeneous boundary conditions. A system of differential equations with respect to the velocity potential and flow function is reduced to the standard form. The unknown functions are represented by the sum of holomorphic functions and generalized analytic functions.

The velocity potential and the flow function ψ are the functions of only cylindrical coordinates x and y, and the domain occupied by the moving liquid on an arbitrarily chosen meridional half-space we denote by $S(z)$, where $z = x + iy$. On the planes $\omega_0(z) = \varphi_0(x, y) + i\psi_0(x, y)$, $w = \omega_0'(z)/z'(s)$, the domain $S(z)$ is associated, respectively, with $S(\omega_0'(z))$, $S(w)$ ([1, 2]).

As is known, the functions $\varphi(x,y)$ and $\psi(x,y)$ satisfy the following equations:

$$\frac{\partial\varphi}{\partial x}=\frac{1}{y}\,\frac{\partial\psi}{\partial y}=v_x,\quad \frac{\partial\varphi}{\partial y}=-\frac{1}{y}\,\frac{\partial\psi}{\partial x}=v_y, \tag{1}$$

where v_x and v_y are the projections of velocity onto the axes x and y. Omitting from (1) by turns φ and ψ, we obtain

$$\Delta\varphi+\frac{1}{y}\,\frac{\partial\varphi}{\partial y}=0,\quad \Delta\psi-\frac{1}{y}\,\frac{\partial\psi}{\partial y}=0, \tag{2}$$

where Δ is the Laplace operator ([1, 2]).

Recall that the hydrodynamical problem is assumed to be solved if any of the two functions $\varphi(x,y)$ or $\psi(x,y)$ is known. Besides the equations, for their definition we have the following boundary conditions. The normal velocity on a free surface and on the body surface is equal to zero, $\frac{\partial\varphi}{\partial n}=0$, where n is the normal directed into the liquid. The flow function ψ on the body surface and on a free surface is constant, $\psi=const$. The last condition, $\psi=const$, is equivalent to the condition $\frac{\partial\varphi}{\partial n}=0$ for φ. The constant in the condition $\psi=const$ may take different values on different boundaries. On the x-axis we can put $\psi=0$. But the difference of values ψ on the flow surfaces is equal to the liquid discharge between these surfaces, divides by 2π, and hence on the tube walls (if the tube lays on the boundary) $\psi=\pi v_\infty n^2/(2\pi)$, where n is the tube radius and v_∞ is the accumulating from the left flow at infinity. A form of free surfaces is unknown beforehand, but there is the supplementary condition of constancy of the velocity modulus v, equivalent to the condition of pressure constancy. This condition can be written in the form $\frac{1}{y^2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]=\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2=v_0^2$ [1, 2], where v_0 is equal to v on a free surface. The unknown functions φ and ψ can be represented as follows: $\varphi=y^{-1/2}\varphi_1$, $\psi=y^{1/2}\psi_1$. Then

$$\Delta\varphi_1+\frac{1}{4}\,\varphi_1=0,\quad \Delta\psi_1-\frac{3}{4}\,y^{-1}\psi_1=0. \tag{3}$$

The system (1) is compatible, whereas the system (3) is incompatible.

The half-plane $\mathrm{Im}(\zeta)\ge 0$ (or $\mathrm{Im}(\zeta)$) of the plane $\zeta=\xi+i\eta$ is mapped conformally onto the domains $S(z)$ occupied by the moving liquid, onto the the domains $S(\omega_0)$ and $S(w)$, where $w=\omega_0'(\zeta)/z'(\zeta)$. Thus we obtain

$$\frac{\partial\varphi}{\partial\xi}=\frac{1}{y(\xi,\eta)}\,\frac{\partial\psi}{\partial\eta},\quad \frac{\partial\varphi}{\partial\eta}=-\frac{1}{y(\xi,\eta)}\,\frac{\partial\psi}{\partial\xi}\,. \tag{4}$$

The system (4) can be rewritten as

$$\begin{gathered}\frac{\partial}{\partial\xi}\left(y(\xi,\eta)\,\frac{\partial\varphi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(y(\xi,\eta)\,\frac{\partial\varphi}{\partial\eta}\right)=0,\\ \frac{\partial}{\partial\xi}\left(\frac{1}{y(\xi,\eta)}\,\frac{\partial\psi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{1}{y(\xi,\eta)}\,\frac{\partial\psi}{\partial\eta}\right)=0\end{gathered} \tag{5}$$

or

$$
\begin{aligned}
&\Delta\varphi(\xi,\eta) + \frac{1}{y(\xi,\eta)}\left[\frac{\partial y}{\partial\xi}\frac{\partial\varphi}{\partial\xi} + \frac{\partial y}{\partial\eta}\frac{\partial\varphi}{\partial\eta}\right] = 0,\\
&\Delta\psi(\xi,\eta) - \frac{1}{y(\xi,\eta)}\left[\frac{\partial y}{\partial\xi}\frac{\partial\psi}{\partial\xi} + \frac{\partial y}{\partial\eta}\frac{\partial\psi}{\partial\eta}\right] = 0.
\end{aligned}
\tag{6}
$$

Assuming that there exist analytic (holomorphic) functions $z = f(\zeta) = x(\xi,\eta) + i\psi(\xi,\eta)$, $\zeta = \xi + i\eta$, $\omega_0(\zeta) = \varphi_0(\xi,\eta) + i\psi_0(\xi,\eta)$, $w(\zeta) = \omega'(\zeta)/z'(\zeta)$, we can map conformally $\mathrm{Im}(\zeta) \geq 0$ onto the domains $S(z)$, $S(\omega_0(\zeta))$ and $S(w)$.

It can be easily seen from (1) that $\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y} = 0$. Consequently, the lines $\varphi = const$ and $\psi = const$ are orthogonal, however, the mapping $f(x,y)$ is not conformal.

There take place the equalities $\varphi(x,-y) = \varphi(x,y)$, $\psi(x,-y) = \psi(x,y)$, i.e., φ and ψ are the even functions with respect to y^2. The system (6) can be rewritten as follows:

$$
\begin{aligned}
\varphi(\xi,\eta) = y^{-1/2}\varphi_1(\xi,\eta), \quad \psi(\xi,\eta) = y^{1/2}\psi_1(\xi,\eta);\\
\varphi_1(\xi,\eta) = \varphi_{10}(\xi,\eta) + \varphi_2(\xi,\eta), \quad \psi_1(\xi,\eta) = \psi_{10}(\xi,\eta) + \psi_2(\xi,\eta);
\end{aligned}
\tag{7}
$$

It follows from (7) that

$$
\begin{aligned}
&\Delta\varphi_1(\xi,\eta) + \frac{1}{4}\left|\frac{f'(\zeta)}{y(\xi,\eta)}\right|^2 \varphi_1(\xi,\eta) = 0,\\
&\Delta\psi_1(\xi,\eta) - \frac{3}{4}\left|\frac{f'(\zeta)}{y(\xi,\eta)}\right|^2 \psi_1(\xi,\eta) = 0.
\end{aligned}
\tag{8}
$$

Recall that the system (8) is incompatible. Thus we can write the first equation of (8), bearing in mind (7), as

$$
\varphi_1(\xi,\eta) + \frac{1}{4}\iint\limits_{\mathrm{Im}(\zeta)} G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2 \varphi_2(x_1,y_1)\,dx_1\,dy_1 = \varphi^*_{10}(\xi,\eta), \tag{9}
$$

where

$$
\varphi^*_{10}(\xi,\eta) = -\Bigg\{\varphi_{10}(\xi,\eta) + \frac{1}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x_1,y_1)\left|\frac{f'(x_1+iy_1)}{y(x_1,y_1)}\right|^2 \varphi_{10}(x_1,y_1)\,dx_1\,dy_1\Bigg\}. \tag{10}
$$

The function $y^{-1/2}(\xi,\eta)\varphi_{10}(\xi,\eta)$ and the Green's function G for $\mathrm{Im}(\zeta) \geq 0$ are known. As for the function $\psi_2(\zeta)$, we act analogously. Thus we have obtained the Fredholm integral equation of second kind. The homogeneous equation, corresponding to equation (9), has no characteristic numbers. A solution (9) can be obtained by the method of successive approximations.

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