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THE PRINCIPAL CONTACT PROBLEM FOR THE ELASTIC MIXTURE

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Dedicated to the memory of Professor Tengiz Gegelia on the occasion of his 80th birthday.

Abstract. The principle contact problem is a known problem. The availability and the uniqueness of the solution of this problem in the classic theory of elasticity are studied in [1]-[5] for the space, in [6], [7] for the plane. The principle contact problem in the three-dimensional theory of elastic mixture is examined in [8], [9]. In this paper, the solution of the principle contact problem for the elastic mixture, when the contact line is a circle, is given by absolutely and uniformly convergent series. The uniqueness of solution of this problem is studied.

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Problem. Let the circle S, with the radius R, divide the plane in two domains: inner domain D_0 and outside one D_1. It is supposed that D_0 and D_1 are filled with different elastic mixtures. For the static equation systems [10], [11] of the elastic mixture theory:

$$\begin{aligned} &a_1^j\Delta(u^j)' + b_1^j graddiv(u^j)' + c^j\Delta(u^j)'' + d^j graddiv(u^j)'' = 0,\\ &c^j\Delta(u^j)' + d^j graddiv(u^j)' + a_2^j\Delta(u^j)'' + b_2^j graddiv(u^j)'' = 0. \end{aligned} \tag{1}$$

In each domain D_j $(j = 0,1)$ a regular solution $u^j(x) \in C^2(D_j \bigcap C^1(\bar{D}_j))$ $(\bar{D}_j = D_j \bigcup (S))$ should be found, such that on the circle S it satisfies the contact conditions:

$$u^1(z)^- - u^0(z)^+ = f(z), \quad P^1u^1(z)^- - P^0u^0(z)^+ = F(z), \tag{2}$$

and the regularity conditions in the infinity: $u^j(x) = O(1), r^2\frac{\partial}{\partial(x_k)} = O(1), k = 1,2,$ where $u^j = ((u^j)', (u^j)'')$, $(u^j)'$ and $(u^j)''$ are the particular displacement vectors, $P^ju^j \equiv P^j(\partial_z, n)u^j = ((P^ju^j)', (P^ju^j)''), j = 0,1$; $(P^ju^j)'$and $(P^ju^j)''$ are the particular stress vectors [11], $r^2 = x_1^2 + x_2^2$, $x = (x_1, x_2) \in (D_j)$, $z = (z_1, z_2) \in (S)$, $x = (r,\psi)$, $n = (n_1, n_2), n_i = \dfrac{x_i}{r}, i = 1,2$; $f = (f', f'')$ and $F = (F', F'')$ are functions on S, and (at the same time)

$$\int\limits_S F(y)d_yS = 0, \int\limits_S [y \times F]d_yS = 0, y = (r,\psi).$$

Theorem. *Any two solutions of the principal contact problem of statics elastic mixture theory may differ from each other only by the constant vector.*

We use the Green's formula to prove for D_0 inner and D_1 outer domains:

$$\begin{aligned} \int\limits_{D_0} E_0(v^0, v^0)dx &= \int\limits_{S} (v^0)^+ \cdot (P^0 v^0)^+ dS, \\ \int\limits_{D_1} E_1(v^1, v^1)dx &= -\int\limits_{S} (v^1)^- \cdot (P^1 v^1)^- dS, \end{aligned} \tag{3}$$

where v^j is the difference of any two solutions in D_j, $j = 0,1$. v^j satisfies system (1) and homogeneous conditions $(2)_0$, E_j are the quadratic forms corresponding to the potential energy of the mixture. They are positively defined forms. In the (3), by term to term addition we obtain a new equality, under integral expression in the right part of which may be transformed thus:

$$(v^0)^+ \cdot (P^0 v^0)^+ - (v^1)^- \cdot (P^1 v^1)^- = ((v^0)^+ - (v^1)^-) \cdot (P^0 v^0)^+ + (v^1)^- \cdot [(P^0 v^0)^+ - (P^1 v^1)^-].$$

According to the conditions $(2)_0$ this difference is always equal to zero. We will get, that $E_j(v^j, v^j) = 0$, from where [11]:

$$(v^j)' = p^j \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} + (q^j)', (v^j)'' = p^j \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} + (q^j)'', j = 0,1. \tag{4}$$

By taking into consideration $(2)_0$ we will conclude that: $p^j = 0, (v^j)' = (q^j)', (v^j)'' = (q^j)''$. The theorem is proved.

We seek for solution u^j the following way [12]:

$$\begin{aligned} &(u^j)'(x) = grad\Phi_1^j(x) + r^2 grad[(\xi_1^j + \frac{1}{2})r\frac{\partial}{\partial r} + 2\xi_1^j]\Phi_2^j(x) + \beta_1^j(r\frac{\partial}{\partial r} + 2)\Phi_3^j(x) \\ &-xr\frac{\partial}{\partial r}[(2\xi_1^j - 1)\Phi_2^j(x) + 2\beta_1^j\Phi_3^j(x)] + (\Psi^j)'(x), \\ &(u^j)''(x) = grad\Phi_4^j(x) + r^2 grad\{[(\beta_2^j + \frac{1}{2})r\frac{\partial}{\partial r} + 2\beta_2^j]\Phi_3^j(x) \\ &+\xi_2^j(r\frac{\partial}{\partial r} + 2)\Phi_2^j(x)\} - xr\frac{\partial}{\partial r}[(2\beta_2^j - 1)\Phi_3^j(x) + 2\xi_2^j\Phi_2^j(x)] + (\Psi^j)''(x), \end{aligned} \tag{5}$$

where $\Psi_i^0 = A_i^0 x + B_i^0 \breve{x}$, $\Psi_i^1 = \frac{1}{r^2}[A_i^1 x + B_i^1 \breve{x}]$, $x = (x_1, x_2)$, $\breve{x} = (-x_2, x_1)$, $i = 1,2$, $\Delta_1^j = a_1^j a_2^j - (c^j)^2$, $\xi_1^j = \frac{1}{2\Delta_1^j}(c^j d^j - b_1^j a_2^j - \Delta_1^j)$, $\xi_2^j = \frac{1}{2\Delta_1^j}(c^j b_1^j - a_1^j d^j)$, $\beta_1^j = \frac{1}{2\Delta_1^j}(c^j b_2^j - a_2^j d^j)$, $\beta_2^j = \frac{1}{2\Delta_1^j}(c^j d^j - b_2^j a_1^j - \Delta_1^j)$, Φ_k^j is any unknown harmonic function, $k = 1,2,3,4;\ j = 0,1$. We seek for Φ_k^j in the following way [13]:

$$\begin{aligned} \Phi_k^0(x) &= \sum_{m=0}^{\infty} \left(\frac{r}{R}\right)^m (X_{mk}^0 \cdot \nu_m(\psi)), x \in (D_0), \\ \Phi_k^1(x) &= \sum_{m=0}^{\infty} \left(\frac{R}{r}\right)^m (X_{mk}^1 \cdot \nu_m(\psi)), x \in (D_1), \end{aligned} \tag{6}$$

where X^j_{mk} is a two-component constant vector. To find X^j_{mk} we write the conditions (2) in the tangent and normal components. In the obtained equality we put the (5) and (6), and using expansion in Fourier series, we pass to the boundary, when $r \to R$. We obtain systems of linear algebraic equations with respect to unknown X^j_{mk}, for $j = 0$ and $j = 1$ separately:

$$\sum_{p=1}^{4} a^j_{kp} X^j_{mp} = b^j_{mk}, k = 1, 2, 3, 4; m = 0, 1, ..., \tag{7}$$

where a^j_{kp} and b^j_{mk} are the known two-component vectors, depended on the elastic constant and on the radius R. The determinant of the system (7) does not equal to zero due two the uniqueness of the problem set. Putting values of X^j_{mk} obtained from the systems (7) in the (6) and (5) we get the u^j, regular solutions of the said problem in the form of the infinite series.

As it is well known the properties of these series and off their first derivatives (including the boundary), guarantee to find the functions given on the boundary to satisfy the following conditions: $f \in C^3(S)$, $F \in C^2(S)$.

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REFERENCES

1. Kupradze V.D., Gegelia T.G., Basheleishvili M.O., Burchuladze T.V. Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity, *Nauka, Moscow*, 1976, 664 pp. (in Russian).

2. Kupradze V.D. Methods of a potential in the theory of elasticity, *Fizmatgiz, Moscow*, 1963, 472 pp. (in Russian).

3. Basheleisvili M.O., Gegelia T.G. About basic tree-dimensional BVP for piecewise isotropic bodes, *Dokl. Akad. Nauk. SSSR*, **160**, 1 (1965), 50-53 (in Russian).

4. Natroshvili D.G. The principal contact problem for a sphere, *Sem. of Inst. of Appl. Math.-Abstract of papers*, **8** (1973), 19-22 (in Russian).

5. Fichera G. Existence theorem in the theory elasticity, *Mir, Moscow*, 1974, 157 pp. (in Russian).

6. Basheleishvili M.O. About some plane boundary contact problems, Some problems of the elastic theory, *Proc. Vekua Inst. Appl. Math.*, **10** (1981), 5-21.

7. Basheleishvili M.O. Solution one contact problem of the theory of elasticity, *Sb. nauchn. trudov. TGPI Im. Puskina, Tbilisi*, 1985, 27-35 (in Russian).

8. Arsenishvili G., Natroshvili D. Interface problems of the theory of elastic mixtures, *Rep. Enlarged Session Sem. Vekua Inst. Appl. Math.*, **8**, 2 (1993), 8-11.

9. Dzhagmaidze A.Y. Contact problems in the theory of elastic mixture, *Rep. Enlarged Sessions Sem. Vekua Inst. Appl. Math.*, **2** (1991), 75-78 (in Russian).

10. Green A.E., Steel T.R. Constitutive equations for interacting continua, *Inst.I. Eng. Sci.*, **4** (1966), 483-500.

11. Natroshvili D.G., Dzhagmaidze A.Y., Svanadze M.G. Some problems in the linear theory of elastics mixtures, *Tbilisi University Press, Tbilisi*, 1986, 215 pp. (in Russian).

12. Giorgashvili L.G.,Toradze D.I., Tsagareli I.I. Solution of the first boundary value problems of thermoelastic mixture for a disc. *Rep. Enlarged Session Sem. Vekua Inst. Appl. Math.*, **20**, 1-3 (2005), 32-36.

13. Mikhlin S.G. A course in mathematical physics, *Nauka, Moscow*, 1968, 576 pp. (in Russian).

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