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ON ONE TWO DIMENSIONAL PROBLEM OF STATICS IN THE THEORY OF ELASTIC MIXTURES WITH A PARTIALLY UNKNOWN BOUNDARY

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Abstract. In the present paper we investigate the second two dimensional boundary value problem of statics in the theory of elastic mixtures with a partially unknown boundary for an infinite isotropic elastic plate.

Using the methods of the theory of elastic functions are defined a stressed state of the plate.

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1⁰. The homogeneous equation of statics of the theory of elastic mixture in the complex form is written as [2]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + K \frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0, \quad U = (u_1 + iu_2, u_3 + iu_4)^T, \tag{1}$$

where $u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$ are partial displacements,

$$K = -\frac{1}{2} e m^{-1}, \quad e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \quad m^{-1} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}^{-1}, \quad z = x_1 + ix_2;$$

m_k, e_{3+k}, $k = 1, 2, 3$, are expressed in terms of the elastic constants [2].

In [2] M. Basheleishvili obtained the representations:

$$U = (u_1 + iu_2, u_3 + iu_4)^T = m\varphi(z) + \frac{1}{2} e z \overline{\varphi'(z)} + \overline{\psi(z)}, \tag{2}$$

$$TU = \begin{pmatrix} (Tu)_2 - i(Tu)_1 \\ (Tu)_4 - i(Tu)_3 \end{pmatrix} = \frac{\partial}{\partial s(x)} \Big[(A - 2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu \overline{\psi(z)} \Big], \tag{3}$$

where $\varphi(z) = (\varphi_1, \varphi_2)^T$ and $\psi(z) = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions, $\frac{\partial}{\partial s(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}$, $n = (n_1, n_2)^T$ is an arbitrary unit vector,

$$A = 2\mu m, \quad B = \mu e, \quad \mu \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

μ_k, $k=1,2,3$, are elastic constants [2]; $(Tu)_p$, $p=\overline{1,4}$, are the components of stresses,

$$(Tu)_1 = r'_{11}n_1 + r'_{21}n_2 = (a\theta' + c_0\theta'')n_1 - (a_1\omega' + cw'')n_2 - 2\frac{\partial}{\partial s(x)}(\mu_1 u_2 + \mu_3 u_4),$$
$$(Tu)_2 = r'_{12}n_1 + r'_{22}n_2 = (a\theta' + c_0\theta'')n_2 + (a_1\omega' + cw'')n_1 + 2\frac{\partial}{\partial s(x)}(\mu_1 u_1 + \mu_3 u_3),$$
$$(Tu)_3 = r''_{11}n_1 + r''_{21}n_2 = (c_0\theta' + b\theta'')n_1 - (c\omega' + a_2 w'')n_2 - 2\frac{\partial}{\partial s(x)}(\mu_3 u_2 + \mu_2 u_4),$$
$$(Tu)_4 = r''_{12}n_1 + r''_{22}n_2 = (c_0\theta' + b\theta'')n_2 + (c\omega' + a_2 w'')n_1 + 2\frac{\partial}{\partial s(x)}(\mu_3 u_1 + \mu_2 u_3);$$
$$\theta' = \operatorname{div} u', \quad \theta'' = \operatorname{div} u'', \quad \omega' = \operatorname{rot} u' \quad \omega'' = \operatorname{rot} u'';$$

a_1, a_2, c, $a = a_1 + b_1$, $b = a_2 + b_2$, $c_0 = c + d$, b_1, b_2 and d are elastic constants [2].

Let us now consider the vectors:

$$\overset{(1)}{\tau} = (r'_{11}, r''_{11})^T, \quad \overset{(2)}{\tau} = (r'_{22}, r''_{22})^T, \quad \tau = \tau^{(1)} + \tau^{(2)}, \tag{4}$$
$$\overset{(1)}{\eta} = (r'_{21}, r''_{21})^T, \quad \overset{(2)}{\eta} = (r'_{12}, r''_{12})^T, \quad \eta = \overset{(1)}{\eta} + \overset{(2)}{\eta}, \quad \varepsilon^* = \overset{(1)}{\eta} - \overset{(2)}{\eta}, \tag{5}$$
$$\sigma_n = \begin{pmatrix} (Tu)_1 n_1 + (Tu)_2 n_2 \\ (Tu)_3 n_1 + (Tu)_4 n_2 \end{pmatrix} = \overset{(1)}{\tau}\cos^2\alpha + \overset{(2)}{\tau}\sin^2\alpha + \eta\sin\alpha\cos\alpha, \tag{6}$$
$$\sigma_s = \begin{pmatrix} (Tu)_2 n_1 - (Tu)_1 n_2 \\ (Tu)_4 n_1 - (Tu)_3 n_2 \end{pmatrix} = \frac{1}{2}(\overset{(2)}{\tau} - \overset{(1)}{\tau})\sin 2\alpha + \frac{1}{2}\eta\cos 2\alpha - \frac{1}{2}\varepsilon^*, \tag{7}$$
$$\sigma_t = \begin{pmatrix} [r'_{21}n_1 - r'_{11}n_2, r'_{22}n_1 - r'_{12}n_2]^T S \\ [r''_{21}n_1 - r''_{11}n_2, r''_{22}n_1 - r''_{12}n_2]^T S \end{pmatrix}$$
$$= \overset{(1)}{\tau}\sin^2\alpha + \overset{(2)}{\tau}\cos^2\alpha - \eta\sin\alpha\cos\alpha. \tag{8}$$

Here $n = (n_1, n_2)^T = (\cos\alpha, \sin\alpha)^T$, $S = (-n_2, n_1)^T = (-\sin\alpha, \cos\alpha)^T$ and $\alpha(t)$ is angle between the outer normal to the contour L of the point t and ox_1 axis.

After lengthy but elementary calculation we obtain:

$$\sigma_n + \sigma_t = \tau = 2(2E - A - B)\operatorname{Re}\varphi'(t), \quad \varepsilon^* = 2(A - B - 2E)\operatorname{Im}\varphi'(t), \tag{9}$$
$$\overset{(1)}{\tau} - \overset{(2)}{\tau} - i\eta = 2\Big[B\bar{t}\varphi''(t) + 2\mu\psi''(t)\Big], \tag{10}$$
$$\sigma_n - i\sigma_s = (2E - R)\overline{\varphi'(t)} - B\varphi'(t) + (B\bar{t}\varphi''(t) + 2\mu\psi'(t))e^{2i\alpha}, \tag{11}$$
$$\Big[(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\Big]_L = -i\int_L e^{i\alpha}(\sigma_n + i\sigma_s)ds, \tag{12}$$
$$\sigma_n + 2\mu\Big(\frac{\partial U_s}{\partial s} + \frac{U_n}{\rho_0}\Big) + i\Big[\sigma_s - 2\mu\Big(\frac{\partial U_n}{\partial s} - \frac{U_s}{\rho_0}\Big)\Big] = 2\varphi'(t), \tag{13}$$

where ρ_0^{-1} is the curvature of the contour L at the point t;

$$U_n = \begin{pmatrix} u_1 n_1 + u_2 n_2 \\ u_3 n_1 + u_4 n_2 \end{pmatrix}, \quad U_s = \begin{pmatrix} u_2 n_1 - u_1 n_2 \\ u_4 n_1 - u_3 n_2 \end{pmatrix}. \tag{14}$$

2^0. Let us consider the stressed state of an infinite isotropic elastic plate denoted by D. Suppose that the plate is weakened by holes of equal size which are located periodically with period 2π on the ox_1-axis which at the same time is the axis of symmetry. The boundary of each hole consists of rectilinear segments L_1 lying on straight lines $x_2 = \pm b^0$ and of unknown smooth contours L_2. We assumed that the angle size at angular points of the boundary does not exceed $\pi/2$. Suppose that on the plate at infinity there takes place one-sided, contracting, constant stress parallel to the ox_2-axis, and rotation is absent, i. e.

$$\overset{(1)}{\tau}{}^{\infty} = 0, \quad \overset{(2)}{\tau}{}^{\infty} = q, \quad \overset{(1)}{\eta}{}^{\infty} + \overset{(2)}{\eta}{}^{\infty} = 0. \tag{15}$$

Also suppose that

$$U_n = U_0 = \text{const}, \quad \sigma_s = 0 \quad \text{on} \quad L_1, \tag{16}$$

$$\sigma_n = \sigma_s = 0, \quad \sigma_t = K^0 = \text{const} \quad \text{on} \quad L_2. \tag{17}$$

The problem is stated as follows [4]: Find a stressed state of the body and an unknown contour L_2 such that the vector σ_t takes constant value on L_2.

By virtue of formulas (3), (9), (12) and (13) the boundary conditions (16)

$$\operatorname{Im}\Big[(A-2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\Big] = 0, \quad \operatorname{Im}\varphi'(t) = 0 \quad \text{on} \quad L_1, \tag{18}$$

$$\begin{gathered} \operatorname{Re}\varphi'(t) = \frac{1}{2}(2E-A-B)^{-1}K^0, \\ (A-2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)} = B^0 \quad \text{on} \quad L_2, \end{gathered} \tag{19}$$

where $\varphi(z)$ and $\psi(z)$ satisfy the definite conditions [4], K^0 and B^0 are constants which will be defined during solving the problem.

It is proved that

$$\begin{gathered} \varphi(z) = \frac{1}{2}(2E-A-B)^{-1}qz, \quad K^0 = q, \\ \psi(z) = Hz + O(z^{-1}), \quad H = -\frac{1}{4}\mu^{-1}q, \quad z \in D. \end{gathered} \tag{20}$$

On the basis of the $(20)_1$, conditions $(18)_1$ and $(19)_2$ take the form

$$\operatorname{Im}(Ht + \overline{\psi(t)}) = 0 \quad \text{on} \quad L_1; \quad Ht + \overline{\psi(t)} = \frac{1}{4}\mu^{-1}B^0 = B^*, \quad \text{on} \quad L_2. \tag{21}$$

Note that if $t \in L_1$ then [1] $\operatorname{Re} e^{-i\alpha(t)}t = \operatorname{Re} e^{-i\alpha(t)}A_k$, where A_k are the affixes of angular points $L = L_1 + L_2$, $\alpha(t) = \pm\frac{\pi}{2}$.

As is mentioned above, we consider the periodic problem. Therefore it is sufficient to investigate the problem for the strip $|x_1| \le h$, $-\infty < x_2 < \infty$. Using the conformal mapping $z = \frac{ih}{2\pi}\ln\zeta^0$, we pass to the plane $\zeta^0 = \xi_1^0 + i\xi_2^0$, cut along the ray $\xi_1^0 \ge 0$. In this case the straight lines $x_1 = \pm h$ will be transformed respectively into the rays $\xi_2 = \pm 0$ and the hole boundary $L = L_1 + L_2$ into an unknown contour $\Gamma = \Gamma_1 + \Gamma_2$.

From the conditions of the problem it follows that on the boundary $x_1 = \pm h$ we have $\sigma_n = \sigma_s = 0$, therefore the $\varphi\Big(\frac{ih}{2\pi}\ln\zeta^0\Big)$ and $\psi\Big(\frac{ih}{2\pi}\ln\zeta^0\Big)$ are continuously extendable along the ray $\xi_1^0 \ge 0$.

Taking this fact into account, the domain considered in the plane ζ is, in fact, the infinite domain D^0 with a hole whose boundary is $\Gamma = \Gamma_1 + \Gamma_2$.

Let the function $\zeta^0 = \omega(\zeta)$ map the unit circle in the plane ζ into the domain D^0. From formulas (21) we have

$$\operatorname{Im}\Big(\frac{ih}{2\pi} H \ln\omega(\sigma) + \overline{\psi\Big(\frac{ih}{2\pi}\ln\omega(\sigma)\Big)}\Big) = 0, \quad \text{on} \quad l_1 = \Big(\frac{ih}{2\pi}\ln(L_1)\Big)^{-1}$$

$$\frac{ih}{2\pi} H \ln\omega(\sigma) + \overline{\psi\Big(\frac{ih}{2\pi}\ln\omega(\sigma)\Big)} = B^*, \quad \text{on} \quad l_2 = \Big(\frac{ih}{2\pi}\ln(L_2)\Big)^{-1}.$$

Besides the above conditions we obviously have $\operatorname{Re}\frac{h}{2\pi}\ln\omega(\sigma) = \pm b^0$, on l_1.

Differentiating the above conditions with recpect to θ^0 and taking into account the fact that $\dfrac{d\sigma}{d\theta^0} = \dfrac{de^{i\theta^0}}{d\theta^0} = ie^{i\theta_0} = i\sigma$ we obtain

$$\operatorname{Im}\Big(H\frac{\omega'(\sigma)}{\omega(\sigma)}\sigma + \overline{\psi_0'(\sigma)}\frac{1}{\sigma}\Big) = 0, \quad \operatorname{Im}\frac{\omega'(\sigma)}{\omega(\sigma)}\sigma = 0, \quad \text{on} \quad l_1 \tag{22}$$

$$H\frac{\omega'(\sigma)}{\omega(\sigma)}\sigma + \overline{\psi_0'(\sigma)}\frac{1}{\sigma} = 0, \quad \text{on} \quad l_2, \tag{23}$$

where $\psi_0'(\sigma) = \psi'\Big(\frac{ih}{2\pi}\ln\omega(\sigma)\frac{\omega'(\sigma)}{\omega(\sigma)}\Big)$.

Let

$$W(\zeta) = \begin{cases} iH\overline{\frac{\omega'(\zeta)}{\omega(\zeta)}}\zeta, & \text{for} \quad |\zeta| < 1 \\ -i\psi_0'\Big(\frac{1}{\zeta}\Big)\overline{\zeta}, & \text{for} \quad |\zeta| > 1. \end{cases} \tag{24}$$

By virtue of (23) the $W(\zeta)$ is analytic on the entire plane ζ cut along the arc l_1 and owing to (22) we get $\operatorname{Re} W^{\pm}(\sigma) = 0$, on l_1.

Consequently, $W(\zeta)$ is the solution of the Dirichet problem for the plane ζ, cut along arc l_1 and the solution is given by the formula [3]

$$W(\zeta) = H(\gamma\zeta + \delta)\Big(\prod_{k=1}^{4}(\zeta - a_k)\Big)^{-\frac{1}{2}}, \quad a_k = \Big(\frac{ih}{2\pi}\ln\omega(a_k)\Big)^{-1}.$$

Having known the $W(\zeta)$, we can define the $\psi_0(\zeta)$ and $\omega(\zeta)$ by formula (24) and hence the stressed state of the body and the equation of the unknown part of the boundary L_2.

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