Seminar of I. Vekua Institute of Applied Mathematics REPORTS, Vol. 34, 2008

GENERAL REPRESENTATION OF SOLUTIONS OF EQUATION OF RADIATION TRANSFER OF POLARIZED LIGHT

Shulaia D.

I. Vekua Institute of Applied Mathematics
of Iv. Javakhishvili Tbilisi State University

Abstract. The Chandrasekhar's equation describing the scattering of polarized light in the case of a combination of Rayleigh and isotropic scattering with arbitrary photon survival probability in an elementary scattering is considered. A theorem on the expansion of the general solution of the equation in terms of eigenvectors of discrete and continuous spectra of the corresponding characteristic equation is represented. The Green's function for infinity domain is constructed.

Keywords and phrases: Basis, eigenfunctions, spectral integral.

AMS subject classification (2000): 45B05, 45E05.

This paper is devoted to the presentation of the method of solving the boundary-value problems for vector equation of Radiation Transfer of the Polarized Light [1]

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu)+I(\tau,\mu)=\frac{1}{2}\omega Q(\mu)\int_{-1}^{1}Q^T(\mu')I(\tau,\mu')d\mu' \tag{1}$$

$$\tau\in(-\infty,+\infty),\quad \mu\in(-1,+1),$$

where $Q(\mu)$ is the square matrix

$$Q(\mu)=\frac{3(c+2)^{1/2}}{2(c+2)}\left\|\begin{matrix} c\mu^2+\frac{3}{2}(1-c) & (2c)^{1/2}(1-\mu^2) \\ \frac{1}{3}(c+2) & 0, \end{matrix}\right\|$$

$\omega\in(0,1)$ is the probability in an elementary scattering, $c\in(0,1)$ is a parameter which characterize the degree of the deviation from the law of scattering of Rayleigh. The symbol T denotes the transpose.

The translational symmetry of Eq.(1) suggests looking for solution of the form

$$I(\tau,\mu)=\exp(-\tau/\nu)\psi_\nu(\mu)$$

which gives the following characteristic equation

$$(\nu-\mu)\psi_\nu(\mu)=\frac{\omega\nu}{2}Q(\mu)\int_{1}^{+1}Q^T(\mu')\psi_\nu(\mu')d\mu'. \tag{2}$$

The values of ν for which Eq.(2) has nonzero solutions are the eigenvalues of the equation. The set of all eigenvalues will be denoted by $S[\nu]$. The discrete spectrum of Eq.(2) consist of two real points $\{\pm\nu_0\}$ which correspond to the two eigenfunctions (see e.g. [2])

$$\psi_{\pm\nu_0}(\mu) = \left\| \begin{array}{c} \psi^{(1)}_{\pm\nu_0}(\mu) \\ \psi^{(2)}_{\pm\nu_0}(\mu) \end{array} \right\|.$$

There is continuum of values of ν, namely $-1 \le \nu \le 1$, for which Eq.(2) has a solution in the distributional sense: (cf. [3])

$$\psi_\nu(\mu) = \frac{\omega\nu}{2}(\nu-\mu)^{-1}M(\nu,\mu) + \delta(\nu-\mu)\left(E - \frac{\omega\nu}{2}\int_{-1}^{+1}(\nu-\mu')^{-1}M(\nu,\mu')d\mu'\right)$$

where

$$M(\nu,\mu) = Q(\mu)Q^T(\nu) + \frac{\omega\nu}{2}Q(\mu)K(\nu)Q^T(\nu)\left(E - \frac{\omega\nu}{2}K(\nu)\right)^{-1},$$

$$K(\nu) = \int_{-1}^{+1}(Q^T(\mu') - Q^T(\nu))(\nu-\mu')^{-1}Q(\mu')d\mu',$$

E is the unit matrix and δ is the Dirac function.

The eigenfunctions of Eq.(2) obey the following orthogonality condition

$$\int_{-1}^{+1}\mu\psi_\nu(\mu)\psi_{\nu'}(\mu)d\mu = D(\nu,\nu')N(\nu)$$

where

$$D(\nu,\nu') = \begin{cases} 0, & \text{when } \nu \neq \nu' \\ \delta(\nu-\nu'), & \text{when either } \nu \text{ or } \nu' \text{ are continuous} \\ \delta_{ij}, & \text{when both } \nu = \nu_i, \ \nu' = \nu_j \text{ are discrete.} \end{cases}$$

$N(\nu)$ is the normalization matrix and δ_{ij} is the Kronecker symbol.

The set of eigenfunctions is complete and hence it obeys the relation

$$\mu\int_{S[\nu]}\psi_\nu(\mu)d\Gamma(\nu)\psi_\nu(\mu') = \delta(\mu-\mu')E$$

where the integral on the left-hand side is the spectral integral and

$$d\Gamma(\nu) = \begin{cases} N^{-1}(\nu)d\nu, & \text{when } \nu \text{ is a continuum eigenvalue} \\ \sum\limits_j \dfrac{\delta(\nu-\nu_j)}{N(\nu_j)}d\nu, & \text{when } \nu \text{ is not a continuum eigenvalue,} \end{cases}$$

here, the sum on the right-hand side is over all discrete eigenvalues.

Theorem. *Every solution of Eq.(1), satisfying the condition H^* (Muskhelishvili class) with respect to μ and differentiable with respect to τ admits the representation*

$$I(\tau,\mu)=\int_{S[\nu]}\exp(-\tau/\nu)\psi_\nu(\mu)U(\nu)d\nu,$$

$$\tau\in(-\infty,+\infty),\quad \mu\in(-1,+1),$$

where the column matrix $U\in H^*$.

Now we are able to construct the Green's function of infinity boundary value problems.

The Green's function for infinite domain is defined as follows. It is required to find vanishing in infinity continuous in $(-\infty,0)\cup(0,+\infty)$ the solution $I(\tau,\mu)$ of (1) satisfying the condition

$$\mu(I(0^+,\mu)-I(0^-,\mu))=F(\mu),\qquad \mu\in(-1,+1)$$

where $F\in H^*$.

By the above mentioned formulas the unique solution $I(\tau,\mu)$ of the formulated problem can be represented in the form

$$I(\tau,\mu)=\pm\int_{S_\pm[\nu]}\exp(\tau/\pm\nu)\psi_\nu(\mu)d\Gamma(\nu)\int_{-1}^{+1}\psi_\nu(\mu')\mu'F(\mu')d\mu'$$

where upper signs are taken when $\tau>0$, and lower signs when $\tau<0$, moreover $S_+[\nu]=\{+\nu_0\}\cup(0,1)$ and $S_-[\nu]=\{-\nu_0\}\cup(-1,0)$.

REFERENCES

1. Chandrasekhar S. Radiative transfer, *New York*, 1960.

2. Latyshev A.V., Moiseev A.V. The boundary-valued problem for the equations of radiation transfer of polarized light, *Fundam. Prikl. Mat.*, **8**, 1 (2002), 97-115 (in Russian).

3. Shulaia D., Sharashidze, N.A Spectral representation of the multigroup transport problem, *Trans. Theory Stat. Phys.* **33**, 2 (2004), 183-202.

Received: 24.06.2008; revised: 22.10.2008; accepted: 24.11.2008.