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ON THE ACCURACY OF AN ITERATION METHOD WHEN SOLVING A SYSTEM OF TIMOSHENKO EQUATIONS

Peradze J.

Iv. Javakhishvili Tbilisi State University

Abstract. A numerical algorithm is constructed for the solution of a nonlinear system of equations which characterizes the dynamic state of a beam. A nonlinear system of algebraic equations is obtained as a result of the discretization with respect to spatial and time variables. As different from [1], where a discrete system is solved by the Picard iteration method, here use is made of the Jacobi iteration method. The error of the method is estimated.

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Let us consider the following system of equations

$$\begin{gathered}\frac{\partial^2 w}{dt^2}(x,t)=\left(cd-a+b\int\limits_0^1\left(\frac{\partial w}{\partial x}(x,t)\right)^2dx\right)\frac{\partial^2 w}{dx^2}(x,t)-cd\frac{\partial\psi}{\partial x}(x,t),\\ \frac{\partial^2\psi}{dt^2}(x,t)=c\frac{\partial^2\psi}{dx^2}(x,t)-c^2d\left(\psi(x,t)-\frac{\partial w}{\partial x}(x,t)\right),\\ 0<x<1,\ 0<t\le T,\end{gathered}\tag{1}$$

with the initial-boundary conditions

$$\begin{gathered}\frac{\partial^p w}{\partial t^p}(x,0)=w^{(p)}(x),\ \ \frac{\partial^p\psi}{\partial t^p}(x,0)=\psi^{(p)}(x),\\ w(0,t)=w(1,t)=0,\ \ \frac{\partial\psi}{\partial x}(0,t)=\frac{\partial\psi}{\partial x}(1,t)=0,\\ p=0,1,\ 0\le x\le 1,\ 0\le t\le T.\end{gathered}\tag{2}$$

System (1) was obtained in [2] when describing a dynamic beam using Timoshenko's theory [3]. For the mechanical constants in (1) we have

$$a,b,c,d>0,\ \ cd-a>0.\tag{3}$$

The question of the solvability of problem (1), (2) is studied in [4] and [1]. In this paper we consider the method of its solution.

A solution of problem (1), (2) will be sought in the form

$$w_n(x,t)=\sum_{i=1}^n w_{ni}(t)\sin i\pi x,\ \ \psi_n(x,t)=\sum_{j=0}^n\psi_{nj}(t)\cos j\pi x,\tag{4}$$

where the coefficients $w_{ni}(t)$ and $\psi_{nj}(t)$ are defined by the Galerkin system

$$\begin{gathered} w''_{ni}(t)+\left(cd-a+b\frac{\pi^2}{2}\sum_{l=1}^{n}l^2w_{nl}^2(t)\right)\pi^2i^2w_{ni}(t)-cd\pi i\psi_{ni}(t)=0,\\ \psi''_{nj}(t)+c\pi^2j^2\psi_{nj}(t)+c^2d(\psi_{nj}(t)-\pi jw_{nj}(t))=0,\ w_{n0}(t)=0,\ 0<t\le T, \end{gathered} \tag{5}$$

$$\begin{gathered} \frac{d^pw_{ni}}{dt^p}(0)=2\int_0^1 w^{(p)}(x)\sin i\pi x dx,\quad \frac{d^p\psi_{nj}}{dt^p}(0)=2\int_0^1\psi^{(p)}(x)\cos j\pi x dx,\\ p=0,1,\quad i=1,2,\dots,n,\quad j=0,1,\dots,n. \end{gathered} \tag{6}$$

Using these equalities, the function $\psi_{n0}(t)$ is immediately defined to be equal to

$$\psi_{n0}(t)=2\left(\cos c\sqrt{d}\,t\int_0^1\psi^{(0)}(x)dx+\frac{1}{c\sqrt{d}}\sin c\sqrt{d}\,t\int_0^1\psi^{(1)}(x)dx\right).$$

Therefore it is assumed that the indexes i and j in system (5), (6) coincide and take the values $1,2,\dots,n$.

To solve problem (5),(6) on the time interval $[0,T]$ we introduce the grid with step $\tau=T/M$ and nodes $t_m=m\tau$, $m=0,1,\dots,M$. The approximate values of $w_{ni}(t_m)$ and $\psi_{ni}(t_m)$ denoted by w_{ni}^m and ψ_{ni}^m will be sought by means of the scheme

$$\begin{gathered} w_{ni\bar{t}t}^{m-1}+\frac{1}{4}\sum_{l=0}^{1}\Bigg[\left(cd-a+b\frac{\pi^2}{2}\frac{\sum_{j=1}^n j^2\left(\left(w_{nj}^{m-l}\right)^2+\left(w_{nj}^{m-l-1}\right)^2\right)}{2}\right)\\ \times\pi^2i^2\left(w_{ni}^{m-l}+w_{ni}^{m-l-1}\right)-cd\pi i\left(\psi_{ni}^{m-l}+\psi_{ni}^{m-l-1}\right)\Bigg]=0, \end{gathered} \tag{7.1}$$

$$\begin{gathered} \psi_{ni\bar{t}t}^{m-1}+\frac{1}{4}c\sum_{l=0}^{1}\left[\left(cd+\pi^2i^2\right)\left(\psi_{ni}^{m-l}+\psi_{ni}^{m-l-1}\right)\right.\\ \left.-cd\pi i\left(w_{ni}^{m-l}+w_{ni}^{m-l-1}\right)\right]=0,\\ i=1,2,\dots,n,\quad m=2,3,\dots,M. \end{gathered} \tag{7.2}$$

To calculate w_{ni}^p and ψ_{ni}^p, $p=0,1$, we use the following formulas obtained from (5)

$$\begin{gathered} w_{ni}^p=w_{ni}(0)+p\Bigg\{\tau w'_{ni}(0)-\frac{\tau^2}{2}\Bigg[\left(cd-a+b\frac{\pi^2}{2}\sum_{j=1}^n j^2w_{nj}^2(0)\right)\pi^2i^2w_{ni}(0)\\ -cd\pi i\psi_{ni}(0)\Bigg]\Bigg\},\\ \psi_{ni}^p=\psi_{ni}(0)+p\left\{\tau\psi'_{ni}(0)-\frac{\tau^2}{2}\left[c\left(cd+\pi^2i^2\right)\psi_{ni}(0)-c^2d\pi iw_{ni}(0)\right]\right\},\\ p=0,1,\quad i=1,2,\dots,n. \end{gathered}$$

Let us transform system (7). From (7.2) we obtain the equality

$$\psi_{ni}^{m}=-\psi_{ni}^{m-2}+\left(\frac{1}{\tau^2 c}+\frac{cd+\pi^2 i^2}{4}\right)^{-1}\left[\left(\frac{2}{\tau^2 c}-\frac{cd+\pi^2 i^2}{2}\right)\psi_{ni}^{m-1}\right.$$
$$\left.+\frac{cd\pi i}{4}\sum_{l=0}^{1}\left(w_{ni}^{m-l}+w_{ni}^{m-l-1}\right)\right],\quad i=1,2,\ldots,n, \tag{8}$$

by means of which (7.1) implies

$$\frac{8}{\tau^2\pi^2 i^2}w_{ni}^{m}+\left[2(cd-a)-\frac{c^2d^2}{2}\left(\frac{1}{\tau^2 c}+\frac{cd+\pi^2 i^2}{4}\right)^{-1}\right.$$
$$\left.+b\frac{\pi^2}{2}\sum_{j=1}^{n}j^2\left(\left(w_{nj}^{m}\right)^2+\left(w_{nj}^{m-1}\right)^2\right)\right]\left(w_{ni}^{m}+w_{ni}^{m-1}\right)=\frac{8}{\tau^2\pi^2 i^2}f_{ni}^{m}, \tag{9}$$
$$i=1,2,\ldots,n,$$

where

$$f_{ni}^{m}=2w_{ni}^{m-1}-w_{ni}^{m-2}-\frac{\tau^2}{8}\left[2(cd-a)-\frac{c^2d^2}{2}\left(\frac{1}{\tau^2 c}+\frac{cd+\pi^2 i^2}{4}\right)^{-1}\right.$$
$$\left.+b\frac{\pi^2}{2}\sum_{j=1}^{n}j^2\left(\left(w_{nj}^{m-1}\right)^2+\left(w_{nj}^{m-2}\right)^2\right)\right]\pi^2 i^2\left(w_{ni}^{m-1}+w_{ni}^{m-2}\right) \tag{10}$$
$$+d\pi i\left(\frac{1}{\tau^2 c}+\frac{cd+\pi^2 i^2}{4}\right)\psi_{ni}^{m-1}.$$

We will solve system (9) layer-by-layer using the nonlinear Jacobi iteration method. This means that having an approximation for w_{ni}^{l} and ψ_{ni}^{l}, $0\le l<m$, we consider (9) as a system of equations with respect to the unknowns w_{ni}^{m}, $i=1,2,\ldots,n$, which are defined by the iteration process

$$\frac{8}{\tau^2\pi^2 i^2}w_{ni,k+1}^{m}+\left\{2(cd-a)-\frac{c^2d^2}{2}\left(\frac{1}{\tau^2 c}+\frac{cd+\pi^2 i^2}{4}\right)^{-1}\right.$$
$$\left.+b\frac{\pi^2}{2}\left[i^2\left(\left(w_{ni,k+1}^{m}\right)^2+\left(w_{ni}^{m-1}\right)^2\right)+\sum_{\substack{j=1\\ j\ne i}}j^2\left(\left(w_{nj,k}^{m}\right)^2+\left(w_{nj}^{m-1}\right)^2\right)\right]\right\} \tag{11}$$
$$\times\left(w_{ni,k+1}^{m}+w_{ni}^{m-1}\right)=\frac{8}{\tau^2\pi^2 i^2}f_{ni}^{m},$$
$$k=0,1,\ldots,\quad i=1,2,\ldots,n,$$

where $w_{ni,k+p}^{m}$ denotes the $(k+p)$-th iteration approximation for w_{ni}^{m}, $p=0,1$. For simplicity, here and in the sequel we do not take into account the errors in w_{ni}^{m-l} and ψ_{m}^{m-l}, $l=1,2$, and f_{ni}^{m}, $i=1,2,\ldots,n$.

As for approximations for ψ_{ni}^{m}, by virtue of (8) the $(k+1)$-th iteration approximation $\psi_{ni,k+1}^{m}$ is defined by the formula

$$\begin{aligned} \psi_{ni,\,k+1}^{m} = -\psi_{ni}^{m-2} + \left(\frac{1}{\tau^2 c} + \frac{cd+\pi^2 i^2}{4}\right)^{-1} \Bigg[\left(\frac{2}{\tau^2 c} - \frac{cd+\pi^2 i^2}{2}\right)\psi_{ni}^{m-1} \\ + \frac{cd\pi i}{4}\left(w_{ni,\,k+1}^{m} + 2w_{ni}^{m-1} + w_{ni}^{m-2}\right)\Bigg], \quad i = 1,2,\ldots,n. \end{aligned} \tag{12}$$

Since (11) and (10) do not contain $\psi_{ni,l}^{m}$, $0 \le l \le k$, formula (12) can be used at every m and i only once after the final approximation $w_{ni,k+1}^{m}$ at the m-th layer has already been found.

For fixed i, (11) is a cubic equation with respect to $w_{ni,\,k+1}^{m}$. Applying to it the Cardano formula [5], we write

$$iw_{ni,k+1}^{m} = -\frac{1}{3} iw_{ni}^{m-1} + \sigma_{i,1} - \sigma_{i,2}, \quad i = 1,2,\ldots,n, \tag{13}$$

where

$$\begin{aligned} \sigma_{i,p} &= \left[(-1)^p \frac{s_i}{2} + \left(\frac{s_i^2}{4} + \frac{r_i^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}, \quad p = 1,2, \\ r_i &= e_i + \frac{2}{3}\left(iw_{ni}^{m-1}\right)^2 + \frac{16}{\tau^2 b\pi^4 i^2}, \\ s_i &= \frac{2}{3} iw_{ni}^{m-1}\left(e_i + \frac{10}{9}\left(iw_{ni}^{m-1}\right)^2\right) - \frac{16}{\tau^2 b\pi^4 i^2}\left(\frac{iw_{ni}^{m-1}}{3} + if_{ni}^{m}\right), \\ e_i &= \frac{1}{b\pi^2}\left[4(cd-a) - c^2 d^2\left(\frac{1}{\tau^2 c} + \frac{cd+\pi^2 i^2}{4}\right)^{-1}\right] \\ &\quad + \sum_{\substack{j=1 \\ j\ne i}} j^2\left(\left(w_{nj,\,k}^{m}\right)^2 + \left(w_{nj}^{m-1}\right)^2\right). \end{aligned} \tag{14}$$

System (13) can be represented as

$$\begin{aligned} im_{ni,\,k+1}^{m} = g_i(1w_{n1,k}^{m}, 2w_{n2,k}^{m}, \ldots, nw_{nn,k}^{m}), \\ i = 1,2,\ldots,n. \end{aligned} \tag{15}$$

To find a condition for the convergence of process (15) and to estimate its error, we consider the Jacobi matrix

$$J = \left(\frac{\partial g_i}{\partial (jw_{nj,\,k}^{m})}\right)_{i,j=1}^{n}. \tag{16}$$

By virtue of (13)-(15) we see that on the principal diagonal of the matrix J we have zeros. As to nondiagonal elements, $i \neq j$, we have for them

$$\begin{aligned}\frac{\partial g_i}{\partial(jw^m_{nj,\,k})} &= -\frac{1}{9}jw^m_{nj,\,k}\sum_{p=1}^{2}\frac{1}{\sigma^2_{i,\,p}}\Bigg[2iw^{m-1}_{ni}\\ &\quad +(-1)^p\left(s_i iw^{m-1}_{ni}+\frac{1}{3}r_i^2\right)\left(\frac{s_i^2}{4}+\frac{r_i^3}{27}\right)^{-\frac{1}{2}}\Bigg].\end{aligned}$$

If to this equality we apply the formulas obtained from (14)

$$\sigma_{i,\,1}\sigma_{i,\,2}=\frac{r_i}{3},\quad \sigma^3_{i,\,2}-\sigma^3_{i,\,1}=s_i,\quad \left(\frac{s_i^2}{4}+\frac{r_i^3}{27}\right)=\frac{\sigma^3_{i,\,1}+\sigma^3_{i,\,2}}{2},$$

then we obtain

$$\begin{aligned}\frac{\partial g_i}{\partial(jw^m_{nj,\,k})} &= -\frac{4}{9}ijw^{m-1}_{ni}\,w^m_{nj,\,k}\left(\sigma^2_{i,\,1}-\frac{r_i}{3}+\sigma^2_{i,\,2}\right)^{-1}+\frac{2}{3}jw^m_{nj,k}s_i\\ &\quad\times\left(\sigma^4_{i,\,1}+\frac{r_i^2}{9}+\sigma^4_{i,\,2}\right)^{-1},\quad i\neq j.\end{aligned}\tag{17}$$

Assume that for ∇ we have $0 < \nabla \leq 4$ and the grid step τ be chosen so that the condition

$$c^2d^2\left(\frac{4}{\tau^2 c}+cd+\pi^2 i^2\right)^{-1} < cd-a+\frac{4-\nabla}{\tau^2\pi^2 i^2},\qquad i=1,2,\ldots,n,\tag{18}$$

is fulfilled.

Let us describe the way how to fulfill this condition. Note that if the parameters of the problem satisfy the inequality $a\left(\frac{1}{\pi^2}+\frac{1}{cd}\right) < 1$, then according to (18) we do not have to impose any restriction on τ. In the general case, in order that condition (18) be fulfilled exactly, we must solve the inequality which follows from it at every $i = 1, 2, \ldots, n$ and is biquadratic with respect to τ. Among all the values obtained, a minimal τ will be the desired value. Note that when the Galerkin method is applied, the value n is not as a rule large in decompositions like (4), which means that the approach we are discussing does not need much effort. The difficulty can be avoided if (18) is replaced by the requirement $\tau < (4-\nabla)^{\frac{1}{2}}/cd$ which guarantees the fulfilment of (18) or if we set $\nabla = 4$ in (18). In the latter case it is sufficient to solve one elementary inequality which follows from (18) for $i = 1$.

Further, we use (17) together with (14) and (18) and also take into account the fact that for the function

$$u^p(\xi) = (-r)^p + \sum_{l=0}^{1}\left[\xi + (-1)^l(\xi^2+r^3)^{\frac{1}{2}}\right]^{\frac{2}{3}p},$$
$$-\infty < \xi < \infty,\quad r = const > 0,\quad p = 1, 2,$$

we have the equality $\min|u^{(p)}(\xi)| = u^{(p)}(0) = (2p-1)r^p$. As a result, having denoted $\omega = \max(a, cd-a)$, for $i \neq j$ we obtain

$$\left|\frac{\partial g_i}{\partial(jw^m_{nj,k})}\right| \leq \frac{1}{3\nabla^2}\tau^2 b\pi^4 i^3 j|w^m_{nj,k}|\Bigg\{\tau^2 b\pi^4 i^2 |w^{m-1}_{ni}|\Bigg[\omega\frac{1}{b\pi^2} + \sum_{l=1}^{n} l^2\left(\frac{1}{4}\left(w^m_{nl,k}\right)^2 + \frac{5}{18}\left(w^{m-1}_{nl}\right)^2\right)\Bigg] + (2+\nabla)|w^{m-1}_{ni}| + 6|f^m_{ni}|\Bigg\}. \tag{19}$$

Let us introduce into consideration the matrix norm $||A|| = \max\limits_{1\leq j\leq n}\sum\limits_{i=1}^{n}|a_{ij}|$ where $A = (a_{ij})^n_{i,j=1}$. In addition to this, we take into account that [6]

$$\sum_{i=1}^{n} i^{2p} = \frac{n(n+1)(2n+1)}{6}\left(\frac{3n^2+3n-1}{5}\right)^{p-1}, \qquad p = 1, 2. \tag{20}$$

Let us estimate the norm of the matrix J defined by equality (16). By (19) and (20) we get

$$||J|| \leq \frac{1}{3\nabla^2}\tau^2 b\pi^4\big(\max_{1\leq j\leq n} j|w^m_{nj,k}|\big)\frac{n(n+1)(2n+1)}{6}\Bigg\{\tau^2 b\pi^4\frac{3n^2+3n-1}{5} \times\sum_{i=1}^{n} i|w^{m-1}_{ni}|\Bigg[\omega\frac{1}{b\pi^2} + \sum_{i=1}^{n} i^2\left(\frac{1}{4}\left(w^m_{ni,k}\right)^2 + \frac{5}{18}\left(w^{m-1}_{ni}\right)^2\right)\Bigg] +(2+\nabla)\sum_{i=1}^{n} i|w^{m-1}_{ni}| + 6\sum_{i=1}^{n} i|f^m_{ni}|\Bigg\}.$$

For some q, $0 < q < 1$, let us define the domain

$$\left\{v = (v_i)^n_{i=1} \in R^n : \sum_{i=1}^{n} i|w^m_{ni,0} - v_i| \leq \frac{1}{1-q}\sum_{i=1}^{n} i|w^m_{ni,0} - w^m_{ni,1}|\right\} \tag{21}$$

and assume that in this domain we have $||J|| \leq q$ and that all $w^m_{n,k} = (w^m_{ni,k})^n_{i=1}$, $k = 0, 1, \ldots$, belong to it. For this it is sufficient that

$$\tau \leq \left[\frac{-\beta + (\beta^2 + 4\alpha\gamma)^{\frac{1}{2}}}{2\alpha}\right]^{\frac{1}{2}},$$

where

$$\alpha = b\pi^4\frac{3n^2+3n-1}{5}\sum_{i=1}^{n} i|w^{m-1}_{ni}|\Bigg[\omega\frac{1}{b\pi^2} + \frac{1}{4}\Bigg(\sum_{i=1}^{n} i\Big(|w^m_{ni,0}| + \frac{1}{1-q} \times|w^m_{ni,0} - w^m_{ni,1}|\Big)\Bigg)^2 + \frac{5}{18}\sum_{i=1}^{n}\left(iw^{m-1}_{ni}\right)^2\Bigg], \ \beta = \sum_{i=1}^{n} i\big((2+\nabla)|w^{m-1}_{ni}| + 6 \times|f^m_{ni}|\big), \ \gamma = \frac{18q\nabla^2}{b\pi^4 n(n+1)(2n+1)}\left[\sum_{i=1}^{n} i\Big(|w^m_{ni,0}| + \frac{1}{1-q}|w^m_{ni,0} - w^m_{ni,1}|\Big)\right]^{-1}.$$

In that case, as follows from the principle of contruction mappings [7], system (9) has a unique solution $w_n^m = (w_{ni}^m)_{i=1}^n$ in domain (21), the iteration process (11) converges, $\lim\limits_{k\to\infty} w_{ni,k}^m = w_{ni}^m$, $i = 1, 2, \ldots, n$, and also the estimate

$$\sum_{i=1}^n i|w_{ni,k}^m - w_{ni}^m| \le \frac{q^k}{1-q} \sum_{i=1}^n i|w_{ni,0}^m - w_{ni,1}^m|$$

is fulfilled for the error.

Taking this estimate together with (12) into account, we conclude that process (12) also converges, $\lim\limits_{k\to\infty} \psi_{ni,k}^m = \psi_{ni}^m$, where ψ_{ni}^m is described by formula (8). The error is estimated by the inequality

$$\sum_{i=1}^n i|\psi_{ni,k}^m - \psi_{ni}^m| \le \rho\tau \frac{q^k}{1-q} \sum_{i=1}^n i|w_{ni,0}^m - w_{ni,1}^m|,$$

where $\rho = \dfrac{1}{2}\sqrt{cd}\left(\dfrac{4}{c^2d} + \tau^2\right)^{-\frac{1}{2}}$.

The problems considered here for one- and two-dimensional Kirchhoff equations are studied in [8] and [9].

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