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ON THE INTEGRATION OF THE DIFFERENTIAL SYSTEM OF EQUATIONS FOR NONLINEAR AND NON-SHALLOW SHELLS

Meunargia T.

I. Vekua Institute of Applied Mathematics
of Iv. Javakhishvili Tbilisi State University

Abstract. In this paper the geometrically nonlinear and non-shallow shells are considered. Here under non-shallow shells will be meant three-dimensional shell-type elastic bodies satisfying the conditions $|hb_\alpha^\beta| \le q < 1$ $(\alpha, \beta = 1, 2)$, in contrast to shallow shells, for which the assumption $hb_\alpha^\beta \cong 0$ is accepted, where h is the semi-thickness, b_α^β are mixed components of the curvature tensor of the shell's midsurface S.

Using the method I. Vekua [1] and the method of a small parameter [2] two-dimensional system of equations for the nonlinear and non-shallow shells is obtained [3]. For any approximation of order N the complex representations of the general solutions are obtained.

Keywords and phrases: Non-shallow shells, metric tensor and tensor of curvature, midsurface of the shell.

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1. Making use of vector and tensor notations, the equilibrium equation of the 3-D elastic bodies and stress-strain relations can be written as follows:

$$\begin{gathered}\nabla_i \boldsymbol{\sigma}^i + \boldsymbol{\Phi} = 0, \quad \boldsymbol{\sigma}^i = \tau^{ij}(\mathbf{R}_j + \partial_j \mathbf{u}), \quad \tau^{ij} = E^{ijpq} e_{pq} \\ E^{ijpq} = \lambda g^{ij} g^{pq} + \mu(g^{ip} g^{jq} + g^{iq} g^{jp}) \quad (\mathbf{R}_j = \partial_j \mathbf{R}, \quad g^{ij} = \mathbf{R}^i \mathbf{R}^j), \\ e_{pq} = \frac{1}{2}(\mathbf{R}_p \partial_q \mathbf{u} + \mathbf{R}_q \partial_p \mathbf{u} + \partial_p \mathbf{u} \partial_q \mathbf{u}), \quad (i, j, p, q = 1, 2, 3)\end{gathered} \tag{1}$$

where ∇_i are covariant derivatives with respect to the space coordinates x^i, $\boldsymbol{\sigma}^i$ are covariant constituents of the stress tensor, $\boldsymbol{\Phi}$ is vector of volume forces. τ^{ij} and e_{pq} are contravariant and covariant components of the stress and strain tensors, $\mathbf{u}$ is the displacement vector, $\mathbf{R}_i$ and $\mathbf{R}^i$ are the covariant and contravariant basis vector of space.

Basis vectors $\mathbf{R}_i$ $(\mathbf{R}^i)$ are expressed by formulas [1]:

$$\begin{gathered}\mathbf{R}_\alpha = (a_\alpha^\beta - x_3 b_\alpha^\beta)\mathbf{r}_\beta, \quad \mathbf{R}^\alpha = \vartheta^{-1}[(1 - 2Hx_3)a_\alpha^\beta + x_3 b_\alpha^\beta]\mathbf{r}^\beta, \\ \mathbf{R}_3 = \mathbf{R}^3 = \mathbf{n} \ \ (a_\alpha^\beta = \mathbf{r}_\alpha \mathbf{r}^\beta),\end{gathered} \tag{2}$$

where $\mathbf{r}_\alpha$ $(\mathbf{r}^\alpha)$ and $\mathbf{n}$ are, respectively, the basis vectors and the normal of the midsurface S, $2H = b_\alpha^\alpha$, $K = b_1^1 b_2^2 - b_2^1 b_1^2$, $\vartheta = 1 - 2Hx_3 + Kx_3^2$, $(-h \leqslant x_3 = x^3 \leqslant h)$.

The relation (1) can be written as [3]:

$$\frac{1}{\sqrt{a}} \frac{\partial \sqrt{a} \vartheta \boldsymbol{\sigma}^\alpha}{\partial x^\alpha} + \frac{\partial \vartheta \boldsymbol{\sigma}^3}{\partial x^3} + \vartheta \boldsymbol{\Phi} = 0, \tag{3}$$

$$\boldsymbol{\sigma}^i = A^i_{i_1} M^{i_1 j_1 p_1 q_1} [A^p_{p_1}(\mathbf{r}_{q_1}\partial_p \mathbf{u}) + A^q_{q_1}(\mathbf{r}_{p_1}\partial_q \mathbf{u}) + A^p_{p_1} A^q_{q_1} \partial_p \mathbf{u} \partial_q \mathbf{u}](\mathbf{r}_{j_1} + A^j_{j_1}\partial_j \mathbf{u}),$$

where a is the discriminant of metric tensor of the midsurface S,

$$M^{i_1 j_1 p_1 q_1} = \lambda a^{i_1 j_1} a^{p_1 q_1} + \mu(a^{i_1 p_1} a^{j_1 q_1} + a^{i_1 q_1} a^{j_1 p_1}), \tag{4}$$

$$A^\alpha_{\alpha_1} = \mathbf{R}^\alpha \mathbf{r}_{\alpha_1}, \quad A^\alpha_3 = \mathbf{R}^\alpha \mathbf{n} = 0, \quad a^{\alpha\beta} = \mathbf{r}^\alpha \mathbf{r}^\beta, \quad a^{\alpha 3} = a^{3\alpha} = \mathbf{r}^\alpha \mathbf{n} = 0, \quad a^{33} = 1.$$

2. The two-dimensional finite system of equilibrium equations with respect to component of displacement vector in the isometric coordinates has the complex form [3]:

$$\begin{aligned}
&4\mu\partial_{\overline{z}}\Big(\Lambda^{-1}\partial_z \overset{(m,n)}{u_+}\Big) + 2(\lambda+\mu)\partial_{\overline{z}} \overset{(m,n)}{\theta} + 2\lambda\partial_{\overline{z}} \overset{(m,n)}{u'_3} - \\
&(2m+1)\mu\Big[2\partial_{\overline{z}}\Big(\overset{(m-1,n)}{u_3} + \overset{(m-3,n)}{u_3} + \cdots\Big) + \overset{(m-1,n)}{u'_+} + \overset{(m-3,n)}{u'_+} + \cdots\Big] = \overset{(m,n)}{F_+}, \\
&\mu\Big(\nabla^2 \overset{(m,n)}{u_3} + \overset{(m,n)}{\theta'}\Big) - (2m+1)\Big[\lambda\Big(\overset{(m-1,n)}{\theta} + \overset{(m-3,n)}{\theta} + \cdots\Big) \\
&+(\lambda+2\mu)\Big(\overset{(m-1,n)}{u'_3} + \overset{(m-3,n)}{u'_3} + \cdots\Big)\Big] = \overset{(m,n)}{F_3}
\end{aligned} \tag{5}$$

$$\Big(m = 0,1,\ldots,N; \ \overset{(k,n)}{u_i} = 0, \ k > N, \ \overset{(m,n)}{u_+} = \overset{(m,n)}{u_1} + i\overset{(m,n)}{u_2}\Big),$$

where

$$\begin{aligned}
&\overset{(m)}{u_i} = \frac{2m+1}{2h}\int_{-h}^{h} u_i P_m\Big(\frac{x_3}{h}\Big)dx_3 = \sum_{n=1}^{\infty} \overset{(m,n)}{u_i}\varepsilon^n, \\
&\overset{(m)}{u'_i} = (2m+1)\Big(\overset{(m+1)}{u_i} + \overset{(m+3)}{u_i} + \cdots\Big), \\
&\overset{(m,n)}{\theta} = 2\,\mathrm{Re}\,\Big(\Lambda^{-1}\partial_z \overset{(m,n)}{u_+}\Big), \ \ z = x^1 + ix^2, \ \ 2\partial_z = \partial_1 - i\partial_2, \\
&\nabla^2 = 4\Lambda^{-1}\partial^2_{z\overline{z}}\,, \ \ ds^2 = \Lambda(z,\overline{z})dz\,d\overline{z},
\end{aligned}$$

P_m is the Legendre polynomial of degree m, the right-hand sides $\overset{(m)}{F_+} = \overset{(m)}{F_1} + i\overset{(m)}{F_2}$ and $\overset{(m)}{F_3}$ are expressed by means of previous approximations, i.e.,

$$\overset{(m)}{F} = \sum_{n=1}^{\infty} \overset{(m,n)}{F}\varepsilon^n, \qquad \overset{(m,n)}{F} = F\big(\overset{(m,n-1)}{u}, \overset{(m,n-2)}{u}, \ldots\big).$$

Here $\varepsilon = \frac{h}{R}$ is small parameter, R is a certain characteristic radius of curvature of the midsurface of shell [2].

Below the upper index will be omitted.

The solution of the homogeneous system (5) we can find the form

$$\overset{(m)}{u}_+ = \partial_{\overline{z}}\Big(\overset{(m)}{V_1} + i\overset{(m)}{V_2}\Big) + \Big(\frac{1}{\pi}\iint\limits_S \frac{\overline{\varphi'(\zeta)} - æ_1\varphi'(\zeta)dS_\zeta}{\overline{\zeta} - \overline{z}} - \overline{\psi'(\zeta)}\Big)\delta_{0m}$$

$$-\Big(\frac{1}{\pi}\iint\limits_S \frac{\Phi'(\zeta) + \overline{\Phi'(\zeta)}dS_\zeta}{\overline{\zeta} - \overline{z}} + \eta_1\overline{\Phi''(z)} - 2\overline{\Psi'(z)}\Big)\delta_{1m} + æ_2\overline{\varphi''(z)}\delta_{2m} + \eta_2\overline{\Phi''(z)}\delta_{3m},$$

$$\overset{(m)}{u_3} = \overset{(m)}{V_3} - \Big(\frac{1}{\pi}\iint\limits_S (\Phi'(\zeta) + \overline{\Phi'(\zeta)})ln|\zeta - z|dS_\zeta - \Psi(z) - \overline{\Psi(z)}\Big)\delta_{0m} \qquad (6)$$

$$-\frac{3}{2}æ_2\Big[(\varphi'(z) + \overline{\varphi'(z)})\delta_{1m} + (\Phi'(z) + \overline{\Phi'(z)})\delta_{2m}\Big],$$

$$\overset{(0)}{V_1} = \overset{(0)}{V_2} = 0, \;\; \overset{(0)}{u_3} = \Psi(z) + \overline{\Psi(z)}, \quad if\ N = 0,$$

$$\delta_{ij} = \begin{cases} 0, \ i \neq j, \\ 1, \ i = j, \end{cases} \quad (dS_\zeta = \Lambda(\zeta, \overline{\zeta})d\zeta d\overline{\zeta}, \ \zeta = \xi + i\eta).$$

where $\varphi'(z)$,$\psi'(z)$,$\Phi'(z)$ and $\Psi'(z)$ are holomorphic functions of z and express the biharmonic solution of the system (5),

$$æ_1 = \begin{cases} \dfrac{\lambda + 3\mu}{\lambda + \mu}, \ N = 0, \\ \dfrac{5\lambda + 6\mu}{3\lambda + 2\mu}, \ N \neq 0, \end{cases} \qquad \eta_1 = \begin{cases} \dfrac{\lambda + 2\mu}{\mu}, \ N = 1, \\ \dfrac{4(\lambda + \mu)}{\lambda + 2\mu}, \ N = 2, \\ \dfrac{23\lambda + 24\mu}{5(\lambda + 2\mu)}, \ N = 3, \end{cases}$$

$$æ_2 = \frac{4\lambda}{3(3\lambda + 2\mu)}, \quad \eta_2 = -\frac{4(3\lambda + 4\mu)}{15(\lambda + 2\mu)}.$$

Substituting expressions (6) in (5) we have

$$(\lambda + 2\mu)\nabla^2\overset{(m)}{V_1} + 2\lambda\overset{(m)}{V_3'} - (2m+1)\Big[2\Big(\overset{(m-1)}{V_3} + \overset{(m-3)}{V_3} + \cdots\Big) + \overset{(m-1)}{V_1'} + \overset{(m-3)}{V_1'} + \cdots\Big] = 0,$$

$$\mu\nabla^2\Big(\overset{(m)}{V_3} + \frac{1}{2}\overset{(m)}{V_1'}\Big) - (2m+1)\Big[\lambda\nabla^2\Big(\overset{(m-1)}{V_1} + \cdots\Big) + (\lambda + 2\mu)\Big(\overset{(m-1)}{V_3'} + \cdots\Big)\Big] = 0, \qquad (7)$$

$$\mu\Big[\nabla^2\overset{(m)}{V_2} - (2m+1)\Big(\overset{(m-1)}{V_2'} + \overset{(m-3)}{V_2'} + \cdots\Big)\Big] = 0,$$

$$\Big(\overset{(m)}{V'} = (2m+1)\Big[\overset{(m+1)}{V} + \overset{(m+3)}{V} + \cdots\Big]\Big) \quad (m = 0, 1, ..., N = 2k+1).$$

Using now the matrix notation for $\overset{(m)}{V_i}$ from (6) we obtain

$$\nabla^2 V + AV = 0, \quad \nabla^2 \Omega + B\Omega = 0, \tag{8}$$

where V and Ω are culumn-matrices

$$V = \big(\overset{(2)}{V_1}, \overset{(3)}{V_1}, ..., \overset{(N)}{V_1}, \overset{(3)}{V_1}, \overset{(2)}{V_3}, ..., \overset{(N)}{V_3}\big)^T, \qquad \Omega = \big(\overset{(1)}{V_2}, \overset{(2)}{V_2}, ..., \overset{(N)}{V_2}\big)^T,$$

$$A = \{a_{ij}\}_{i,j=1}^{2N-1}, \qquad B = \{b_{ij}\}_{i,j=1}^{N}.$$

Let the numbers $\alpha_1, \alpha_2, ..., \alpha_{2N-1}$ and $\beta_1, \beta_2, ..., \beta_N$ be simple eigenvalues of matrices A and B respectively, and $A_s = \big(\overset{(2)}{A_s}, ..., \overset{(2N)}{A_s}\big)^T$, $B_s = \big(\overset{(1)}{B_s}, ..., \overset{(N)}{B_s}\big)^T$ their eigenvectors.

The general solutions of homogeneous matrix equations (6) have the form

$$V = \sum_{s=1}^{2N-1} A_s w_s, \qquad \Omega = \sum_{s=1}^{N} B_s \omega_s \tag{9}$$

where w_s and ω_s are arbitrary solutions of the following scalar equations

$$\nabla^2 w_s + \alpha_s w_s = 0, \qquad \nabla^2 \omega_s + \beta_s \omega_s = 0.$$

Hence

$$\begin{aligned}
\overset{(m)}{V_1} &= \sum_{s=1}^{2N-1} \overset{(m)}{A_s} w_s \qquad (m = 2, 3, ..., N), \\
\overset{(m)}{V_2} &= \sum_{s=1}^{N} \overset{(m)}{B_s} \omega_s \qquad (m = 1, 2, ..., N), \quad \overset{(0)}{V_2} = 0, \\
\overset{(m)}{V_3} &= \sum_{s=1}^{N} \overset{(m+N)}{A_s} w_s \quad (m = 1, 2, ..., N).
\end{aligned} \tag{10}$$

Making use of equations (6) for $m = 0$ and $m = 1$ we obtain

$$\begin{aligned}
\overset{(0)}{V_1} &= \frac{2\lambda}{\lambda + 2\mu} \sum_{s=1}^{2N-1} \frac{1}{\alpha_s} \Big(\overset{(N+1)}{A_s} + \overset{(N+3)}{A_s} + \cdots + \overset{(2N)}{A_s}\Big) w_s, \qquad \overset{(0)}{V_2} = 0, \\
\overset{(0)}{V_3} &= -\frac{1}{2} \sum_{s=1}^{2N-1} \Big[\overset{(3)}{A_s} + \cdots + \overset{(N)}{A_s} + \frac{3\lambda}{\lambda + 2\mu} \frac{1}{\alpha_s} \Big(\overset{(N+2)}{A_s} + \cdots + \overset{(2N-1)}{A_s}\Big)\Big] w_s, \\
\overset{(1)}{V_1} &= \frac{6\lambda}{\lambda + 2\mu} \sum_{s=1}^{2N-1} \frac{1}{\alpha_s} \Big(\overset{(N+2)}{A_s} + \cdots + \overset{(2N-1)}{A_s}\Big) w_s.
\end{aligned} \tag{11}$$

By substituting (9),(10) into (5) we obtain general representations for the components of the displacement vector ($N = 2k+1$)

$$\overset{(0)}{u}_+ = \partial_{\overline{z}} \left\{ \frac{2\lambda}{\lambda+2\mu} \sum_{s=1}^{2N-1} \frac{1}{\alpha_s} \Big(\overset{(N+1)}{A_s} + \cdots + \overset{(2N)}{A_s} \Big) w_s \right.$$

$$\left. - \frac{2}{\pi} \iint\limits_S (æ_1 \varphi'(\zeta) - \overline{\varphi'(\zeta)}) ln|\zeta - z| dS_\zeta - \overline{\Psi(z)} \right\},$$

$$\overset{(0)}{u}_3 = -\frac{1}{2} \sum_{s=1}^{2N-1} \Big[\overset{(3)}{A_s} + \cdots + \overset{(N)}{A_s} + \frac{3\lambda}{\lambda+2\mu} \frac{1}{\alpha_s} \Big(\overset{(N+2)}{A_s} + \cdots + \overset{(2N-1)}{A_s} \Big) \Big] w_s$$

$$+2Re\Big[\Psi(z) - \frac{1}{\pi} \iint\limits_S \Phi'(\zeta) ln|\zeta - z| dS_\zeta \Big], \tag{12}$$

$$\overset{(1)}{u}_+ = \partial_{\overline{z}} \left\{ \frac{6\lambda}{\lambda+2\mu} \sum_{s=1}^{2N-1} \frac{1}{\alpha_s} \Big(\overset{(N+2)}{A_s} + \cdots + \overset{(2N-1)}{A_s} \Big) + i \sum_{s=1}^{N} \overset{(1)}{B_s} \omega_s \right.$$

$$\left. + \frac{2}{\pi} \iint\limits_S (\Phi'(\zeta) + \overline{\Phi'(\zeta)}) ln|\zeta - z| dS_\zeta - \eta_1 \overline{\Phi'(z)} - 2\overline{\Psi(z)} \right\},$$

$$\overset{(m)}{u}_+ = \partial_{\overline{z}} \Big\{ \sum_{s=1}^{2N-1} \overset{(m)}{A_s} w_s + i \sum_{s=1}^{N} \overset{(m)}{B_s} \omega_s + æ_2 \overline{\varphi''(z)} \delta_{2m} + \eta_2 \overline{\Phi''(z)} \delta_3 m \Big\},$$

$$\overset{(m)}{u}_3 = \sum_{s=1}^{N} \overset{(m+N)}{A_s} w_s - \frac{3}{2} æ_2 \Big[(\varphi'(z) + \overline{\varphi'(z)}) \delta_{1m} + (\Phi'(z) + \overline{\Phi'(z)}) \delta_{2m} \Big]$$

$$(m = 1, 2, ..., N, \quad N = 2k+1).$$

Let us now express by Riemann functions the particular solutions $\hat{V}$ and $\overset{\wedge}{\Omega}$ of the matrix equations (8)

$$\nabla^2 \hat{V} + A \hat{V} = F, \qquad \nabla^2 \overset{\wedge}{\Omega} + A \overset{\wedge}{\Omega} = L.$$

These solutions have the form

$$\hat{V}(z, \overline{z}) = \frac{1}{u} \int_{z_0}^{z} \int_{\overline{z}_0}^{\overline{z}} \Lambda(t, \overline{t}) R(z, \overline{z}, t, \overline{t}) F(t, \overline{t}) dt d\overline{t}, \tag{13}$$

$$\overset{\wedge}{\Omega}(z, \overline{z}) = \frac{1}{u} \int_{z_0}^{z} \int_{\overline{z}_0}^{\overline{z}} \Lambda(t, \overline{t}) r(z, \overline{z}, t, \overline{t}) L(t, \overline{t}) dt d\overline{t},$$

where $R(z,\overline{z},t,\overline{t})$ and $r(z,\overline{z},t,\overline{t})$ are Riemann functions:

$$R(z,\overline{z},t,\overline{t})=E-\frac{A}{4}\int\limits_{t}^{z}\int\limits_{\overline{t}}^{\overline{z}}\Lambda(t_1,\overline{t})dt_1d\overline{t}_1$$

$$+\Big(\frac{A}{4}\Big)^2\int\limits_{t}^{z}\int\limits_{\overline{t}}^{\overline{z}}\Lambda(t_1,\overline{t}_1)\Big(\int\limits_{t}^{t_1}\int\limits_{\overline{t}}^{\overline{t}_1}\Lambda(t_2,\overline{t}_2)dt_2d\overline{t}_2\Big)dt_1d\overline{t}_1 \tag{14}$$

$$-\Big(\frac{A}{4}\Big)^3\int\limits_{t}^{z}\int\limits_{\overline{t}}^{\overline{z}}\Lambda(t_1,\overline{t}_1)\Big[\int\limits_{t}^{t_1}\int\limits_{\overline{t}}^{\overline{t}_1}\Lambda(t_2,\overline{t}_2)\Big(\int\limits_{t}^{t_2}\int\limits_{\overline{t}}^{\overline{t}_2}\Lambda(t_3,\overline{t}_3)dt_3d\overline{t}_3\Big)dt_2d\overline{t}_2\Big]dt_1d\overline{t}_1+\cdots$$

$$r(z,\overline{z},t,\overline{t})=E-\frac{B}{4}\int\limits_{t}^{z}\int\limits_{\overline{t}}^{\overline{z}}\Lambda(t_1,\overline{t})dt_1d\overline{t}_1$$

$$+\Big(\frac{B}{4}\Big)^2\int\limits_{t}^{z}\int\limits_{\overline{t}}^{\overline{z}}\Lambda(t_1,\overline{t}_1)\Big(\int\limits_{t}^{t_1}\int\limits_{\overline{t}}^{\overline{t}_1}\Lambda(t_2,\overline{t}_2)dt_2d\overline{t}_2\Big)dt_1d\overline{t}_1 \tag{15}$$

$$-\Big(\frac{B}{4}\Big)^3\int\limits_{t}^{z}\int\limits_{\overline{t}}^{\overline{z}}\Lambda(t_1,\overline{t}_1)\Big[\int\limits_{t}^{t_1}\int\limits_{\overline{t}}^{\overline{t}_1}\Lambda(t_2,\overline{t}_2)\Big(\int\limits_{t}^{t_2}\int\limits_{\overline{t}}^{\overline{t}_2}\Lambda(t_3,\overline{t}_3)dt_3d\overline{t}_3\Big)dt_2d\overline{t}_2\Big]dt_1d\overline{t}_1+\cdots$$

Analogous formulas can be written for $N=2k$.

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