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NATURALNESS OF DEFINITION OF THE GENERALIZED STOCHASTIC INTEGRAL IN A BANACH SPACE

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Abstract. In this paper we define the generalized stochastic integral for a wide class of Banach space valued random processes (generalized random processes) with respect to real Wiener processes, which is the generalized random element. If it is decomposable by the random element, then we say that this random element is the stochastic integral. Therefore, the problem of existence of the stochastic integral is reduced to the problem of decomposability of the generalized random element. To show the naturalness of this definition, we consider the case, when the Banach space is C[0; 1]. To this end we introduce the weakly mean square continuous stochastic processes and study some of its properties.

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Introduction

The main problem of developing the stochastic analysis in the Banach space involves the definition of the Ito stochastic integral. Many authors accost to the finite dimensional method, in which the stochastic integral is defined first for step functions, and for the second order stochastic processes as a limit of a sequence of stochastic integrals from step functions converging to an integrand function. However, this method makes it possible to define the stochastic integral in the Banach spaces with special geometric properties (see [2],[3],[4]). In an arbitrary Banach space it is still possible to define the stochastic integral only in case, when the integrand function is nonrandom (see[5],[6]).

In this paper we define the generalized stochastic integral for a wide class of the Banach space valued random processes (generalized random processes) with respect to a real Wiener process, which is the generalized random element. If it is decomposable by the random element, then we say that this random element is the stochastic integral. Therefore, the problem of existence of the stochastic integral is reduced to the problem of decomposability of the generalized random element. This is a natural way of overcoming the present problem. We show this for the case when the Banach space is $C[0,1]$. We introduce also the weakly mean square continuous stochastic processes, and develop some of their behaviours.

Let X be a real separable Banach space, X^* its conjugate, $\mathcal{B}(X)$ the Borel σ-algebra of X, $(\Omega, \mathcal{B}, P)$ a probability space. Continuous linear operator $\mathcal{L} : X^* \to L_2(\Omega, \mathcal{B}, P)$ is called a generalized random element (GRE). Denote by $\mathcal{M}_1 :=$ $= L(X^*, L_2(\Omega, \mathcal{B}, P))$ the Banach space of GRE with the norm $\|\mathcal{L}\| = \sup_{\|x^*\| \le 1} \|\mathcal{L}x^*\|_{L_2}$.

A random element (a measurable map) $\xi : \Omega \to X$ is said to have a weak second order, if for all $x^* \in X^*$, $E\langle \xi, x^*\rangle^2 < \infty$. We can realize the random element ξ as an element of $\mathcal{M}_1$: $\mathcal{L}_\xi x^* = \langle \xi, x^*\rangle$ (continuity of $\mathcal{L}_\xi$ follows from the closed graph theorem). Denote by $\mathcal{M}_2$ the linear space of all random elements of weak second order with the norm $\|\xi\| = \|\mathcal{L}_\xi\|$. Therefore, we can assume $\mathcal{M}_2 \subseteq \mathcal{M}_1$. The correlation operator of $\mathcal{L} \in \mathcal{M}_1$ is defined as $\mathcal{R}_\mathcal{L} : X^* \to X^{**}$, $\mathcal{R}_\mathcal{L} = \mathcal{L}^* L$. $\mathcal{R}_L$ is a positive and symmetric linear operator. If $\mathcal{L} = \mathcal{L}_\xi \in \mathcal{M}_2$, then $\mathcal{R}_\mathcal{L}$ maps X^* to X. It is known (see [1], corollary of lemma 3.1.1), that if $\mathcal{R}$ is a positive and symmetric linear operator from X^* to X, then there exists $(x_k^*)_{k\in N} \subset X^*$ and $(x_k)_{k\in N} \subset X$ such that $\langle \mathcal{R}x_k^*, x_j^*\rangle = \delta_{kj}$, $\mathcal{R}x_k^* = x_k$, and for $x^* \in X^*$, $\mathcal{R}x^* = \sum_{k=1}^{\infty}\langle x_k, x^*\rangle x_k$. Let $\mathcal{L} \in \mathcal{M}_1$. Consider the map $m_\mathcal{L} : X^* \to R^1$, $m_\mathcal{L} x^* = E\mathcal{L}x^*$. $m_\mathcal{L}$ is linear and bounded. Therefore, $m_\mathcal{L} \in X^{**}$, and is called the mean of the GRE $\mathcal{L}$. When $\mathcal{L} \in \mathcal{M}_2$, that is, there exists $\xi : \Omega \to X$ such that $\mathcal{L}x^* = \langle \xi, x^*\rangle$, then $m \in X$ (see[1], Th.2.3.1) and it is the Pettis integral of ξ.

Proposition 1. *Let $\mathcal{L} \in \overline{\mathcal{M}}_2 \subset \mathcal{M}_1$; then $\mathcal{R}_\mathcal{L} : X^* \to X$.*

Proof. Let $\mathcal{L} \in \overline{\mathcal{M}}_2$, then there exists $\xi_n, n = 1, 2, \dots$ such that $\|\mathcal{L}_{\xi_n} - \mathcal{L}\| \to 0$. Denote by $\mathcal{R}_n$, $\mathcal{R}_n : X^* \to X$ the covariance operator of the random element ξ_n. By the factorization lemma $\mathcal{R}_n = \mathcal{L}_{\xi_n}^* \mathcal{L}_{\xi_n}$, $\mathcal{L}_{\xi_n} : X^* \to \mathcal{G}_n$, $\mathcal{L}_{\xi_n}^* : \mathcal{G}_n \to X$ where $\mathcal{G}_n$-is the closed separable subspace of $L_2(\Omega, \mathcal{B}, P)$. Let $\mathcal{H}_2$ denotes $\overline{L(\mathcal{G}_1 \bigcup \mathcal{G}_2)}$, $\mathcal{H}_3 := \overline{L(\mathcal{G}_3 \bigcup \mathcal{H}_2)}, \dots \mathcal{H}_n := \overline{L(\mathcal{G}_n \bigcup \mathcal{H}_{n-1})}$; $\mathcal{H}_2 \subset \mathcal{H}_3 \subset \dots . \mathcal{H} := \bigcup_n \mathcal{H}_n$. For all n we can represent $\mathcal{H}$ as $\mathcal{H}=\mathcal{G}_n \bigoplus \mathcal{G}_n^\perp$. For any $x^* \in X^*$, $\|\mathcal{L}x^* - \mathcal{L}_n x^*\| \to 0$; therefore,

$\mathcal{L}x^* \in \mathcal{H}$ and $\mathcal{R}_\mathcal{L} = \mathcal{L}^*\mathcal{L}. \|\mathcal{R}_n x^* - \mathcal{R}_\mathcal{L} x^*\|_{X^{**}} = \|\mathcal{L}_n^*\mathcal{L}_n x^* - \mathcal{L}^*\mathcal{L}x^*\|_{X^{**}}$

$\le \sup_{\|y^*\|\le 1} |\langle \mathcal{L}_n^*\mathcal{L}_n x^*, y^*\rangle - \langle \mathcal{L}^*\mathcal{L}_n x^*, y^*\rangle|$

$+ \sup_{\|y^*\|\le 1} |\langle \mathcal{L}^*\mathcal{L}_n x^*, y^*\rangle - \langle \mathcal{L}^*\mathcal{L}x^* y^*\rangle| \le \|\mathcal{L}_n x^*\| \sup_{\|y^*\|\le 1} \|\mathcal{L}_n y^* - \mathcal{L}y^*\|$

$+ \sup_{\|y^*\|\le 1} \|\mathcal{L}y^*\| \|\mathcal{L}_n x^* - \mathcal{L}x^*\| \to 0, n \to \infty$. That is $\mathcal{R}_L x^* = \lim_{n\to\infty} \mathcal{R}_n x^* \in X$;

therefore , $\mathcal{R}_L X^* \subseteq X$.

For $C[0, 1]$ we prove below necessary and sufficient condition for $\mathcal{R}_L X^* \subseteq X$.

1. Stochastic integral. Let $(W_t)_{t\in[0,1]}$ be one-dimensional Wiener process, $(\mathcal{F}_t)_{t\in[0,1]}$ an increasing family of σ-algebras such that a)W_t is $\mathcal{F}_t$-measurable for all $t \in [0, 1]$; b) $W_S - W_t$ is independent of the σ algebra $\mathcal{F}_t$ for all $s > t$. $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets in $\mathcal{B}$. Let $(\mathcal{L}_t)_{t\in[0,1]}$ be a family of GRE. We call it a generalized random process (GRP). If we have a weak second-order random process $(\xi_t)_{t\in[0,1]}$, $\xi_t : \Omega \to X$ it will be realized as a GRP-$\mathcal{L}_{\xi_t} x^* = \langle \xi_t, x^*\rangle$.

Definition 1. The GRP $(\mathcal{L}_t)_{t\in[0,1]}$ is called nonanticipating with respect to $(\mathcal{F}_t)_{t\in[0,1]}$, if for all $x^* \in X^*$ the real random process $\mathcal{L}_t x^*$ has a stochastically equivalent measurable modification, and for all $t \in [0, 1]$ $\mathcal{L}_t x^*$ is F_t-measurable. By $T\mathcal{M}_1$ we denote the linear normed spaces of nonanticipating GRP $(\mathcal{L}_t)_{t\in[0,1]}$, for which $\|\mathcal{L}_t\| := \sup_{\|x^*\|\le 1} (\int_0^1 E(\mathcal{L}_t x^*)^2 dt)^{\frac{1}{2}} < \infty$. If $\xi_t : \Omega \to X$, $t \in [0, 1]$ is a random process such that for all $x^* \in X^*$,$\langle \xi_t, x^*\rangle_{t\in[0,1]}$ is nonanticipating, and $\int_o^1 E\langle \xi_t, x^*\rangle^2 dt < \infty$, then, from closed graph theorem it follows that $\mathcal{L}_{\xi_t} \in T\mathcal{M}_1$. For all $x^* \in X^*$ we can define the scalar stochastic integral $\int\limits_0^1 \mathcal{L}_t x^* dW_t$.

Definition 2. For any $(\mathcal{L}_t)_{t\in[0,1]} \in T\mathcal{M}_1$, the operator $I(\mathcal{L}) : X^* \to L_2(\Omega, \mathcal{B}, P)$ defined as $I(\mathcal{L})x^* = \int\limits_0^1 \mathcal{L}_t x^* dW_t$ is called the generalized stochastic integral (GSI) of $(\mathcal{L}_T)_{t\in[0,1]}$.

It is easy to see that $I(\mathcal{L}) \in \mathcal{M}_1$.

Proposition 2. *$I : T\mathcal{M}_1 \to \mathcal{M}_1$ is an isometric operator; for all $(\mathcal{L}_t)_{t\in[0,1]} \in T\mathcal{M}_1$, $EI(\mathcal{L}) = 0 \in X^{**}$, $\langle \mathcal{R}_{I(\mathcal{L})} x^*, y^* \rangle = EI(\mathcal{L})x^* I(\mathcal{L})y^* = \langle \mathcal{R}_{\mathcal{L}} x^*, y^* \rangle$, where $\mathcal{R}_{\mathcal{L}} : X^* \to X^{**}, \langle \mathcal{R}_{\mathcal{L}} x^*, y^* \rangle = \int\limits_0^1 \int_\Omega \mathcal{L}_t x^* \mathcal{L}_t y^* dP dt$.*

Proof. It follows that $E \int\limits_0^1 \mathcal{L}_t x^* dW_t$ is a linear bounded functional from X^* to R^1, that belongs to X^{**}, and equals to 0.

$$\|I(\mathcal{L})\|^2_{\mathcal{M}_1} = \sup_{\|x^*\|\le 1} E(\int\limits_0^1 \mathcal{L}_t x^* dW_t)^2 = \sup_{\|x^*\|\le 1} E \int\limits_0^1 \mathcal{L}_t^2 x^* dt = \|\mathcal{L}\|^2_{T\mathcal{M}_1}$$

The equalities of the proposition follow from the definitions of the correlation operator of a GRE.

Definition 3. A random element $\eta : \Omega \to X$ is called the stochastic integral of $(\mathcal{L}_t)_{t\in[0,1]}$ (if such an element exists), if $\langle \eta, x^* \rangle = I(\mathcal{L})x^*$ almost surely (a.s.) and thereby, we denote it as $\eta = \int\limits_0^1 \mathcal{L}_t dW_t$.

2. The case of $C[0,1]$. We consider the real-valued random processes, $(\xi_t)_{t\in[0,1]}$ which give GRE on $C[0,1]$.

Proposition 3. *Let the random process $(\xi_t)_{t\in[0,1]}$ is such that $\sup_{t\in[0,1]} E(\xi_t)^2 < \infty$, and the realizations of it are a. s. measurable; then, it can realize GRE $T_\xi : C[0,1]^* \to L_2(\Omega, \mathcal{B}, \mathcal{P})$.*

Proof. Let $m(t) := E\xi_t$. It is obvious that $\sup_{t\in[0,1]} |m(t)| < \infty$. Therefore, $m(t) \in M := C[0,1]^{**}, \langle m(t), \varphi \rangle = \int_0^1 m(t) d\varphi(t), \varphi \in C[0,1]^*$, and without loss of generality, we can assume that $m(t) = 0$. Let $r(t,s) := E\xi_t\xi_s$. Then the operator $R_\xi : C[0,1]^* \to M, (R_\xi \varphi)(t) = \int_0^1 r(t,s) d\varphi(s)$ is a positive, symmetric, linear operator. By the factorization lemma (see [1], lemma 3.1.1) $R_\xi = AA^*$, where $A^* : C[0,1]^* \to L_2[0,1]$. Denote by $k(t,\tau)$ the function $A^*\delta_t \in L_2[0,1], \delta_t \in C[0,1]^*, \langle f, \delta_t \rangle = f(t), f \in C[0,1]$. Let G be the Hilbert subspace of $L_2[0,1]$, spanned by $k(t,.), t \in [0,1]$; S-be the Hilbert subspace of $L_2(\Omega, \mathcal{B}, P)$ spanned by $\xi_t, t \in [0,1]$. Then as $r(t,s) = \int_0^1 k(t,\tau)k(s,\tau)d\tau = E\xi_t\xi_s$, $U : G \to S, Uk(t,.) = \xi_t$ is isometry, and $T = UA^* : C[0,1]^* \to L_2(\Omega, \mathcal{B}, P)$ is GRE realized by $(\xi_t)_{t\in[0,1]}, T\delta_t = \xi_t$. We can directly write the operator T_ξ: For any $\varphi \in C[0,1]^*$, let $T_\xi(\varphi) = \int_0^1 \xi_t d\varphi(t)$. This integral exists as if we consider the function $F : ([0,1], \mathcal{B}, \varphi) \to L_2(\Omega, \mathcal{B}, P), F(t) = \xi_t$. Then, for any $\eta \in L_2(\Omega, \mathcal{B}, P), \langle F(t), \eta \rangle = E\xi_t\eta$ is measurable and integrable; therefore the Pettis integral $\int_0^1 \xi_t d\varphi(t) \in L_2(\Omega, \mathcal{B}, P)$

exists and $\sup_{\|\varphi\|\leq 1}\|T_\xi\varphi\|^2 = \sup_{\|\varphi\|\leq 1} E(\int_0^1 \xi_t d\varphi(t))^2 \leq \sup_t E\xi_t^2\|\varphi\|^2$. That is, T is bounded.

R_ξ maps $C[0,1]^*$ to M. When does R_ξ map to $C[0,1]\subset M$? Now we shall answer this question. First we shall introduce the following definition:

Definition 4. The random process $(\xi_t)_{T\in[0,1]}$ is called weakly mean square continuous, if for any t and $t_n \to t, (t_n)_{n\in N}, t$ from $[0,1]$, $\langle \xi_{t_n},\eta\rangle \to \langle \xi_t,\eta\rangle$ when $n\to\infty$ for all $\eta \in L_2(\Omega,\mathcal{B},P)$.

The following theorem is true.

Theorem 1. $(\xi_t)_{t\in[0,1]}$ *is a weakly mean square continuous random process if and only if,* R_ξ *is a continuous operator from* $C[0,1]^*$ *to* $C[0,1]$.

Proof. Let $R_\xi : C[0,1]^* \to C[0,1]$ and S be the Hilbert subspace of $L_2(\Omega,\mathcal{B},P)$ spanned by $(\xi_t)_{t\in[0,1]}$. By the factorization lemma, $R_\xi = TT^*, T^* : C[0,1]^* \to S, T^*\delta_t = \xi_t; L_2(\Omega,\mathcal{B},P) = S\oplus S^\perp$, if $\eta = h\oplus h^\perp, T\eta = Th$. Let $t_n \to t. \langle T^*\delta_{t_n},\eta\rangle = \langle T\eta,\delta_{t_n}\rangle$. As $T\eta \in C[0,1], \langle T^*\delta_{t_n},\eta\rangle \to \langle T^*\delta_t,\eta\rangle$. Therefore, $\langle \xi_{t_n},\eta\rangle \to \langle \xi_t,\eta\rangle$. That is, $(\xi_t)_{t\in[0,1]}$ is weakly mean square continuous.

Let now $(\xi_t)_{t\in[0,1]}$ be a weakly mean square continuous, then $\sup_{t\in[0,1]} E(\xi_t)^2 < \infty$, because if we assume that $\sup_{t\in[0,1]} E(\xi_t)^2 = \infty$, then there exists $(t_k)_{k\in N}$, such that $\sup E(\xi_{t_k})^2 = \infty$. Choose $(t_{k_n})_{k_n\in N}$ converging to some t_0, as $\xi_{t_{k_n}}$ converges weakly in $L_2(\Omega,\mathcal{B},P), \sup_{k_n} E(\xi_{t_{k_n}})^2 < \infty$; therefore, our assumtion is not true, that is, $\sup_{t\in[0,1]} E(\xi_t)^2 < \infty$.

Consider the operator $R_\xi : C[0,1]^* \to M, (R_\xi\varphi)(t) = \int_0^1 r(t,s)d\varphi(s)$, where $r(t,s) = E\xi_t\xi_s$. Let $R_\xi = AA^*$ be the factorization of $R_\xi.A^* : C[0,1]^* \to L_2[0,1]$. Denote $A^*\delta_t$ by $k(t,\tau)\in L_2[0,1]$. $\langle A^*\delta_t, g\rangle = \int_0^1 k(t,\tau)g(\tau)d\tau$. Let G be the Hilbert subspace of $L_2[0,1]$, spanned by $k(t,.), t\in[0,1], S$ be the Hilbert subspace of $L_2(\Omega,\mathcal{B},P)$ spanned by $\xi_t, t\in[0,1]$. The function $k : [0,1]\to L_2[0,1]$ is weakly continuous as if $t_n \to t$, then $\int_0^1 (k(t,\tau)-k(t_n,\tau))g(\tau)d\tau = E(\xi_t - \xi_{t_n})g \to 0, t_n \to t$. For any $g\in L_2[0,1], Ag(t) = \langle Ag,\delta_t\rangle = \langle A^*\delta_t, g\rangle = \int_0^1 k(t,\tau)g(\tau)d\tau. |Ag(t) - Ag(s)| = |\int (k(t,\tau)-k(s,\tau))g(\tau)d\tau| \to 0$, when $s\to t$, as $k : [0,1]\to L_2[0,1]$ is weakly continuous. Therefore, $Ag \in C[0,1]$. That is $R_\xi : C[0,1]^* \to C[0,1]$.

Weakly mean square continuity of random processes is an important condition. For example, for any $L\in[0,1]$ dense subset, if we have random variables $(\xi_s), s\in L$, we can reestablish $(\xi_t)_{t\in[0,1]}$ a.s. as the space $L_2(\Omega,\mathcal{B},\mathcal{P})$ is weakly-sequentially complete. The following theorem is also true.

Theorem 2. *If the random process* $(\xi_t)_{t\in[0,1]}$ *is weakly mean square continuous, then for an arbitrary dense subset* $L\in[0,1]$*, there exists* L*-separable stochastically equivalent modification of* $(\xi_t)_{t\in[0,1]}$.

Proof. Remember that the non-random function $f : [0,1]\to R$ is L-separable, if for all $t\in[0,1]$ $\lim_{\delta\to 0}\sup_{|t-s|<\delta, s\in L} f(s) \geq f(t) \geq \lim_{\delta\to 0}\inf_{|t-s|\leq\delta, s\in L} f(s)$. A random process is L-separable if its realizations are L-separable. Denote $\overline{\xi_t} := \lim_{\delta\to 0}\sup_{|t-s|\leq\delta, s\in L}\xi_s$ and $\underline{\xi_t} := \lim_{\delta\to 0}\inf_{|t-s|\leq\delta, s\in L}\xi_s$ We must prove that $\overline{\xi_t}\geq\xi_t\geq\underline{\xi_t}$ a.s. In order to do we need the following lemma:

Lemma 1. *If the sequence of random variables* $(X_n)_{n\in N}$ *converges weakly mean square to* X*, then* $\lim_{n\to\infty}\sup_{k\geq n} X_k \geq X \geq \lim_{n\to\infty}\inf_{k\geq n} X_k$ *a.s.*

Proof. Without loss of generality assume, that $X = 0$, and prove that

$\lim_{n\to\infty}\sup_{k\geq n} X_k \geq 0$. Let us suppose the contrary. That is for $A := (\omega : \lim_{n\to\infty}\sup_{k\geq n} X_k(\omega) < 0), P(A) > 0$. We can assume that A is the whole Ω. For any $Z \in L_2(\Omega, \mathcal{B}, P)$, we have $\int_\Omega X_n(\omega)Z(\omega)dP \to 0, n \to \infty$, but $\lim_n \sup_{k\geq n} X_k < 0$. Denote $Y_n := \sup_{k\geq n} X_k, Y := \lim_n \sup_{k\geq n} X_k . Y_n \geq Y_{n+1}, n = 1, 2, \dots$ and $Y_n \downarrow Y$ a.s. Therefore, there exists $B \subset \Omega, P(B) > 0$ and such number n, that $Y_n(\omega) < 0$ for $\omega \in B$. As $X_n I_B(\omega) \leq Y_n I_B(\omega)$, there exists the integral $\int_\Omega Y_n(\omega)I_B(\omega)dp$, and $\int_\Omega X_n(\omega)I_B(\omega)dp \leq \int_\Omega Y_n(\omega)I_B(\omega)dp < 0$. $\int_\Omega Y_n(\omega)I_B(\omega)dp$ is a decreasing sequence and $\lim_n \int_\Omega Y_n(\omega)I_B dP < 0$. Therefore, $\lim_{n\to\infty} \int_\Omega X_n(\omega)I_B(\omega)dp \leq \lim_{n\to\infty} \int_\Omega Y_n(\omega)I_B(\omega)dP < 0$, but, as we have weakly mean square continuity, $\int_\Omega X_n(\omega)I_B(\omega)dp \to 0$ Therefore, our supposing the contrary is not true. That is, $\lim_n \sup_{k\geq n} X_k \geq 0$. From the lemma 1 we have, $\overline{\xi_t} \geq \xi_t \geq \underline{\xi_t}$ a.s., which gives the proof of the theorem 2 by the following well known method: Let $\xi'_t(\omega) = \xi_t(\omega)$ if $\overline{\xi_t}(\omega) \geq \xi_t(\omega)$ and $\xi'_t(\omega) = \overline{\xi_t}(\omega)$ for other $\omega \in \Omega$. Then $(\xi'_t)_{t\in[0,1]}$ is stochasticaly equivalent to $(\xi_t)_{t\in[0,1]}$, $(\xi'_s) = (\xi_s)$ for all $s \in L$ a.s. and $\lim_{\delta\to 0}\sup_{|t-s|\leq\delta, s\in L} \xi'_s = \lim_{\delta\to 0}\sup_{|t-s|\leq\delta, s\in L} \xi_s = \overline{\xi_t}$ and $\overline{\xi'_t} \geq \xi'_t \geq \underline{\xi'_t}$. Therefore, $(\xi'_t)_{t\in[0,1]}$ is L-separable.

3. Stochastic integration in $C[0,1]$**.** Let $(\xi_t)_{t\in[0,1]}$ be such that $\sup_s E\xi_t(\omega, s)^2 < \infty$ for all $t \in [0,1]$, then we have a GRP $T_t : C[0,1]^* \to L_2(\Omega, \mathcal{B}, P), T_t\varphi = \int_0^1 \xi_t(\omega, s)d\varphi(s), T_t\delta_s = \xi_t(\omega, s)$. Let $(w_t)_{t\in[0,1]}$ be a areal valued Wiener process, $(\mathcal{F}_t)_{t\in[0,1]}$ be a family of increasing σ-algebra such that a)w_t is $\mathcal{F}_t$-measurable for all $t \in [0,1]$; b) $w_S - w_t$ is independent of the σ algebra $\mathcal{F}_t$ for all $s > t$. $\mathcal{F}_0$ contains all P-null sets in $\mathcal{B}$. Let $\xi_t(\omega, s)$ be nonanticipating and $\sup_s \int_0^1 \int_\Omega \xi_t^2(\omega, s)dpdt < \infty$, then we can define the stochastic integral $\int_0^1 \xi_t(\omega, s)dw_t$. Denote $\eta_s := \int_0^1 \xi_t(\omega, s)dw_t$. The random process $(\eta_s)_{s\in[0,1]}$ gives a GRE on $C[0,1]$, that is $T_\eta : C[0,1]^* \to L_2(\Omega, \mathcal{B}, P), T_\eta(\delta_s) = \int_0^1 \xi_t(\omega, s)dw_t$. Therefore, we define the GSI for the wide class of GRP. The following theorem gives the sufficient condition of existence of the stochastic integral ($C[0,1]$-valued random element)

Theorem 3. *Let $\xi_t(\omega, s)$ be a nonanticipating GRP. If there exists $\alpha > 0, \beta > 0$, $K > 0$ such that $E(\int_0^1 (\xi_t(\omega, s) - \xi_t(\omega, l))^2 dt)^{\alpha/2} \leq K|s - l|^{1+\beta}$, then there exists the stochastic integral $\int_0^1 \xi_t(\omega, s)dw_t$ in $C[0,1]$.*

Proof. Consider the process $\int_0^1 \xi_t(\omega, s)dw_t$. Using Kolmogorov's condition of continuity of the random process, we have $E|\int_0^1 \xi_t(\omega, s)dw_t - \int_0^1 \xi_t(\omega, l)dw_t|^\alpha = E|\int_0^1 (\xi_t(\omega, s) - \xi_t(\omega, l))dw_t|^\alpha \leq C_\alpha E(\int_0^1 (\xi_t(\omega, s) - \xi_t(\omega, l)^2 dt)^{\alpha/2} \leq C_\alpha K|s - l|^{1+\beta}$. Therefore, the process $\eta_s := \int_0^1 \xi_t(\omega, s)dw_t$ has a. s. continuous sample paths. so, this is the random element in $C[0,1]$.

Let now $(W_t)_{t\in[0,1]}$ be a Wiener process with values in $C[0,1]$ adapted to the $(\mathcal{F}_t)_{t\in[0,1]}$. For all $t, s, W_t(s)$ is a Gaussian random variable with mean 0 and correlation function $EW_t(s)W'_t(s') = \min(t, t')r(s, s')$, where $r(s, s')$ is correlation function of a Gaussian process with continuous sample paths. Consider the GSI $\int_0^1 W_t dw_t$. The following statement is true:

Corollary. The stochastic integral $\int_0^1 W_t dw_t$ exists in $C[0,1]$, if there exists $\alpha > 0, \beta > o, K > 0$ such that $(r(t,t) - 2r(t,s) + r(s,s))^{\alpha/2} \leq K|t - s|^{1+\beta}$.

Proof. Denote the GSI $\eta_s := \int_0^1 W_t(s)dw(t)$. Let us now verify the Kolmogorov's condition for η_s: $E|\eta_s - \eta'_s|^\alpha = E|\int_0^1 (W_t(\omega, s) - W_t(\omega, s'))dw_t|^\alpha \leq B_\alpha E(\int_0^1 (W_t(s) - W_t(s'))^2 dt)^{\alpha/2}$. Let us now use the representation of the Wiener process in the Banach space by the a.s. uniformly convergence series (see[7]), $W_t = \sum_{k=1}^\infty \int_0^t e_k(\tau)d\tau\xi_k$, where $(e_k)_{k\in N}, k = 1, 2, \ldots$ is orthonormal basis in $L_2[0, 1]$ and $\xi_k, k = 1, 2, \ldots$ are independent, identically distributed Gaussian random elements in $C[0, 1]$ such that $E\xi = 0$ and $R_\xi = R_{W_1}$. That is $E(\xi_k(s)-\xi_k(s'))^2 = E(W_1(s)-W_1(s'))^2 = r(s, s)-2r(s, s')+r(s', s')$. Therefore, $\xi_k(s) - \xi_k(s') = (r(s, s) - 2r(s, s') + r(s', s'))^{1/2}\gamma_k$, where $\gamma_k, k = 1, 2, \ldots$ are independent, standard Gaussian random variables. Then, it is easy to see that $\sum_{k=1}^\infty \int_0^t e_k(\tau)d\tau(\xi_k(s) - \xi_k(s')) = (r(s, s) - 2r(s, s') + r(s', s'))^{1/2}B_t$, where $B_t, t \in [0, 1]$ is a real-valued Wiener process. Therefore, we have: $E|\eta_s - \eta'_s|^\alpha \leq B_\alpha(r(s, s) - 2r(s, s') + r(s', s'))^{\alpha/2}E(\int_0^1 B_t^2(\omega)dt)^{\alpha/2} \leq K'|s - s'|^{1+\beta}$. Thus, $\eta_s = \int_0^1 W_t(s)dw(t)$ satisfies the Kolmogorov's condition; therefore the stochastic integral as an element of $C[0, 1]$ exists.

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