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SOLUTION TO NON-LOCAL PROBLEM FOR SMOOTH DOMAINS

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Abstract. The nonlocal boundary problem in the three-dimensional space is posed as follows. Let Ω be a simply connected domain belonging to $C^{(2,\alpha)}$, $\partial\Omega$ be the boundary of the domain, and let ω be a simply connected domain belonging to $C^{(2,\alpha)}$ such that $\overline{\omega} \subset \Omega$, $\partial\omega = S \in C^{(2,\alpha)}$. Let further $\zeta = Z(x)$ be a $C^{(2,\alpha)}$-diffeomorphism from $\partial\Omega$ onto S. For any function $f \in C(\partial\Omega)$ we have to find a function $\varphi \in C(\partial\Omega)$ satisfying the boundary condition

$$\varphi(x) - K\varphi(x) = f(x), \quad x \in \partial\Omega,$$

where

$$K\varphi(x) = V(Z(x)),\ \Delta V(x) = 0,\ V(x) = \varphi(x), x \in \partial\Omega.$$

Let $\gamma_1 \in C^{(1,\alpha)}(S)$ be a positive density. Define the density γ_2 as follows

$$U^{\gamma_2}(x) = U^{\gamma_1}(x),\ x \in R^3 \setminus \Omega,\ \gamma_2 \in C^{(1,\alpha)}(\partial\Omega).$$

Theorem. The non-local boundary problem (1) is solvable if and only if $f \in \gamma_2^{\perp}$, where $\gamma_2^{\perp}$ is the annihilator, $\gamma_2^{\perp} \subset C(\partial\Omega)$.

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Non-local boundary problems are considered in [1-6]. The non-local boundary problem in the three-dimensional space is posed as follows. Let Ω be a simply connected domain belonging to $C^{(2,\alpha)}$, $\partial\Omega$ be the boundary of the domain, and let ω be a simply connected domain belonging to $C^{(2,\alpha)}$ such that $\overline{\omega} \subset \Omega, \partial\omega = S \in C^{(2,\alpha)}$. Let further $\zeta = Z(x)$ be a $C^{(2,\alpha)}$- diffeomorphism from $\partial\Omega$ onto S. For any function $f \in C(\partial\Omega)$ we have to find a function $\varphi \in C(\partial\Omega)$ satisfying the boundary condition

$$\varphi(x) - K\varphi(x) = f(x), \quad x \in \partial\Omega, \tag{1}$$

where

$$K\varphi(x) = V(Z(x)) = -\int_{\partial\Omega} \frac{\partial G(Z(x), y)}{\partial \nu_y} \varphi(y) dS_y.$$

Here G is the Green function of the Dirichlet problem for the domain Ω, ν_x is the outer normal, $x \in \partial\Omega$.

Let us introduce necessary definitions.

The Newtonian volume potential and the double layer potential are defined as follows

$$V^g(x) = \int_{\Omega} \Gamma(x, y) g(y) dy, \quad U^{\psi}(x) = \int_{\partial\Omega} \Gamma(x, y)\psi(y) dS_y,$$

$$g \in C(\overline{\Omega}), \quad \psi \in C(\partial\Omega), \quad \Gamma(x,y) = |x-y|^{-1}.$$

Denote by γ_1 the density of the equilibrium potential for $S = \partial\omega$, i.e. $U^{\gamma_1}(x) = 1,\ x \in \overline{\omega},\ \gamma_1 \in C^{(1,\alpha)}(S)$. The density $\gamma_2 \in C^{(1,\alpha)}(\partial\Omega)$ is defined by the equality

$$U^{\gamma_1} = U^{\gamma_2}(x), \quad x \in R^3 \setminus \Omega.$$

Finally let us define the space B_2 of boundary functions:

$$B_2 = \left\{ f : f \in C(\partial\Omega), \quad \int\limits_{\partial\Omega} f(x)\gamma_2(x)dS_x = 0 \right\}.$$

Theorem 1. *The non-local boundary problem* (1) *is solvable if and only if* $f \in B_2$.

Proof. Let us prove first that γ_2 is an eigen-function of the dual (compact) operator K^*. Indeed, by the definition of K we have

$$K^*\gamma_2(y) = -\int\limits_{\partial\Omega} \frac{\partial G(Z(x),y)}{\partial\nu_y}\gamma_2(x)dS_x.$$

It is clear that the function

$$\frac{\partial G(\zeta,y)}{\partial\nu_y}$$

is harmonic with respect to $\zeta \in \Omega$ for any $y \in \partial\Omega$. Beside, the weak sense boundary value [5] is the Dirac measure $\delta_y, y \in \partial\Omega$. This is a consequence of the equality

$$\lim_{\zeta\to y_0} v(\zeta) = -\lim_{\zeta\to y_0}\int\limits_{\partial\Omega} \frac{\partial G(\zeta,y)}{\partial\nu_y}\varphi(y)dS_y = \varphi(y_0) = (\delta_{y_0},\varphi), \quad \varphi \in C(\partial\Omega).$$

According to the balayage principle [5]

$$K^*\gamma_2(y) = -\int\limits_{\partial\Omega} \frac{\partial G(Z(x),y)}{\partial\nu_y}\gamma_2(x)dS_x = (\delta_y,\gamma_2') = (\delta_y,\gamma_2) = \gamma_2(y). \tag{2}$$

Since the support of γ_2 belongs to $\partial\Omega, \gamma_2' = \gamma_2$. Therefore, $K^*\gamma_2 = \gamma_2$.

It is easy to see, by use of the principle of maximum for harmonic functions, that the homogeneous equation $\varphi - K\varphi = 0$ in the space B_2 has only the trivial solution $(\gamma_2(x) > 0, x \in \partial\Omega)$.

According to the Riesz-Schauder theory, the above results imply that the non-local boundary problem (1) is solvable if and only if $f \in B_2$.

Theorem 1 is proved.

Corollary 1. The non-local boundary problem (1) is solvable in the space B_2 if and only if the following equality of potentials holds:

$$\int\limits_S \Gamma(x,\zeta)\gamma_2(Z^{-1}(\zeta))DZ^{-1}(\zeta)dS_\zeta = \int\limits_{\partial\Omega} \Gamma(x,y)\gamma_2(y)dS_y, \ x \in R^3 \setminus \Omega.$$

Corollary 2. Let $\Omega = \{x : |x| < 1\}$. ω be a simply connected domain belonging to $C^{(2,\alpha)}$, $\overline{\omega} \subset \Omega,\ \partial\omega = S,\ \zeta = Z(x) \in C^{(2,\alpha)}$. The non-local boundary problem (1) is solvable in the space

$$B_1 = \left\{ f : f \in C(\partial\Omega), \quad \int\limits_{\partial\Omega} f(x) dS_x = 0 \right\}$$

if and only if $x_0 = (0,0,0) \in \omega$.

According to [6] Corollary 2 is a direct consequence of Theorem 1.

In conclusion let us consider an arbitrary positive density $\gamma_1 \in C^{(1,\alpha)}(S)$, where $S \subset \Omega, S \in C^{(2,\alpha)},\ \Omega \in C^{(2,\alpha)},\ \zeta = Z(x)$ is a $C^{(2,\alpha)}$ - diffeomorfism from $\partial\Omega$ on to S. Define the density γ_2 as follows

$$U^{\gamma_2}(x) = U^{\gamma_1}(x), \quad x \in R^3 \setminus \Omega, \quad \gamma_2 \in C^{(1,\alpha)}(\partial\Omega).$$

It is easy to prove that $K^*\gamma_2 = \gamma_2$. Therefore, the non-local boundary problem (1) is solvable if and only if $f \in \gamma_2^{\perp}$, where $\gamma_2^{\perp}$ is the annihilator, $\gamma_2^{\perp} \subset C(\partial\Omega)$.

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